INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Functional Analysis and Applications Sector

Curriculum in Applied Mathematics

Quasistatic-evolution problems with nonconvex energies: a Young measure approach.

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ACADEMIC YEAR 2007-2008

Il presente lavoro costituisce la tesi presentata da Alice Fiaschi, sotto la direzione del Prof. Gianni Dal Maso, al fine di ottenere l'attestato di ricerca post-universitaria *Doctor Phiosophiae* presso la S.I.S.S.A., Curriculum in Matematica Applicata, Settore di Analisi Funzionale ed Applicazioni. Ai sensi dell'art. 18, comma 3, dello Statuto della Sissa pubblicato sulla G.U. no. 62 del 15.03.2001, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica*.

Trieste, anno accademico 2007-2008.

Contents

Introduction

This thesis is devoted to the study of a generalized notion of quasistatic evolution for a class of rate-independent models in nonlinear elasticity, with nonconvex elastic-energy functional.

The term *rate-independent* characterizes those models which are independent of the rate of the system, in the following sense: if $\boldsymbol{e}(t)$ is a solution of the evolution problem corresponding to a loading $\boldsymbol{l}(t)$, and ϕ is a strictly monotone time-reparametrization, then $(\boldsymbol{e} \circ \phi)(t)$ solves the problem with loading $(\boldsymbol{l} \circ \phi)(t)$. The notion of evolution we are interested in is called *quasistatic*, meaning that we are considering a so slow time scale that the system is assumed to be in equilibrium at each time instant.

A general framework for the study of these problems is the energetic formulation developed by Mielke (see [**33**]). The main advantage of this approach is that it is time-derivative free and therefore allows to deal with cases in which the evolution is not expected to be smooth with respect to time.

The energetic approach to rate-independent models has been widely used in the analysis of many phenomena, like dry friction, elasto-plasticity, delamination or fracture processes, hysteresis in shape-memory alloys, etc.

In this thesis we focus our attention on the mathematical aspects of a model, originally proposed in [**21**], for the study of phase transitions in crystalline materials.

The setting of the problem is as follows: we consider a material whose reference configuration is a bounded region $D \subset \mathbb{R}^d$; the state of the system is determined by the deformation $v\colon D \to \mathbb{R}^N$ and by the internal variable $z\colon D \to Z \subseteq \mathbb{R}^m$, which represents the phase distribution of the material. The stored energy is

$$\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x))\, dx.$$

From a physical point of view, the energy functional should also depend on the temperature, but this dependence is omitted here since this model deals with isothermal transformations[1]. Changes of the phase distribution of the material dissipate an amount of energy that is written as

$$\mathcal{H}(z_{new} - z_{old}) := \int_D H(z_{new}(x) - z_{old}(x))\, dx,$$

[1] The case of temperature-induced phase transformation has been studied in [**34**], [**35**].

where H is a norm on Z, z_{old} is the old phase distribution and z_{new} the new one. Moreover, we require that the admissible deformations satisfy a prescribed time-dependent boundary condition[2] $\boldsymbol{\varphi}(t)$ on ∂D.

Before dealing with the specific case analyzed in this work, let us describe briefly the simpler case of W strictly convex with respect to both variables. An energetic solution of the evolution problem in the time interval $[0,T]$, with initial datum (z_0, v_0), is defined as a pair of functions $(\boldsymbol{z}, \boldsymbol{v})$, depending on time and space, which fulfils the boundary condition and satisfies the following properties:

(1) *stability condition:* for every $t \in [0,T]$ we have

$$\mathcal{W}(\boldsymbol{z}(t), \boldsymbol{v}(t))\rangle \leq \mathcal{W}(\tilde{z}, \tilde{v}) + \mathcal{H}(\tilde{z} - \boldsymbol{z}(t)),$$

for every pair $(\tilde{z}, \tilde{v})$ of functions of x admissible at time t, i.e., satisfying the prescribed boundary condition at time t;

(2) *energy equality:* for every $t \in [0,T]$ we have

$$\mathcal{W}(\boldsymbol{z}(t), \boldsymbol{v}(t)) + \mathrm{Var}_H(\boldsymbol{z}; 0, t) = \\ = \mathcal{W}(z_0, v_0) + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle \, ds,$$

where Var_H is a suitable notion of variation and $\boldsymbol{\sigma}(t)$ is defined by

$$\boldsymbol{\sigma}(t) := \frac{\partial W}{\partial F}(\boldsymbol{z}(t), \nabla \boldsymbol{v}(t)).$$

Clearly, from the energy equality for $(0,t)$, the analogous equality for any interval (s,t) with $0 < s < t$ can be obtained.

These conditions have a natural mechanical interpretation. The stability property guarantees that, unless the boundary condition (or the loading) is modified, it is not energetically convenient any change of the state of the system, if one takes into account the energy dissipated when the state is changed. The variation appearing in the energy equality represents the total energy dissipated in the time interval $(0,t)$, while $\boldsymbol{\sigma}(s)$ is the stress of the system, so that the integral term in the energy equality represents the work due to the change in time of the boundary condition; therefore the energy balance can be regarded as an energy-conservation law.

The main goal in the study of rate-independent problems is providing a constructive proof of the existence of an evolution with the above requisites[3].

To this end, the standard method is based on the study of auxiliary incremental minimum problems, which are used to construct inductively approximate solutions (see [**33**] and references therein). We consider

$$\min \left\{ \mathcal{W}(z, v) + \mathcal{H}(z - \bar{z}) \right\} \tag{1}$$

among all (z, v) admissible at time t, for a given $\bar{z}$. Under suitable regularity and growth assumptions on W and H, the convexity of W guarantees the solvability of this minimum problem and the uniqueness of the minimizer. Fixed a partition $0 = t_0 < \cdots < t_k =$

[2]In this thesis we also consider the more general case of a boundary condition imposed on a subset of the boundary, and the case of a nontrivial external load depending on time $\boldsymbol{l}(t)$ (see Chapter 3, and Chapter 4).

[3]The results about uniqueness of the evolution are still very few, see [**7**], [**39**], [**36**].

T of the time interval, the discrete approximate solution is now constructed with the following inductive process: $(z(t_0), v(t_0))$ coincides with the initial datum (z_0, v_0), and, fixed $(z(t_{i-1}), v(t_{i-1}))$, we define $(z(t_i), v(t_i))$ as the solution of (1) with $t = t_i$ and $\bar{z} = z(t_{i-1})$, for $i = 1, \dots, k$.

The next step consists in the study of the limit of piecewise constant interpolations of $(z(t_i), v(t_i))$ as the mesh size of the partition tends to zero. An a priori bound on these interpolations, provided by the minimality property, allows to extract a convergent subsequence. Using some technical arguments, it is possible to deduce the required stability property and energy balance for the limit of the selected sequence of interpolations[4].

Coming back to our model, the natural stored-energy density W for a multi-phase material has a multi-well structure (see [**40**], [**39**], [**27**], [**21**], [**37**], [**38**]), so that we deal with a density which does not satisfies any convexity assumption with respect to z (in addition, all the results in this thesis are proved without any quasiconvexity assumption with respect to the deformation gradient). This lack of convexity gives rise to many technical difficulties, making the incremental minimum problems (1) unsolvable in usual functional spaces[5]; it is also responsible for the formation of microstructures (see, e.g., [**41**], [**27**], [**37**]). In [**21**], [**4**], [**28**], suitable regularizing spatial terms are introduced in the energy functional to overcome this difficulty, either depending on the gradient of the internal variable, or penalizing phase interfaces.

The aim of this thesis is to study a generalized formulation of the evolution problem avoiding any artificial regularization.

Following the approach proposed in [**12**], we place the problem in a suitable space of Young measures, where the incremental minimum problems can be solved. Then we adapt the standard method of approximate solutions to this extended setting.

To specify the main features of the mathematical setting and highlight the motivations, it is worth pointing out the following facts. Since we assume that W has quadratic growth, the natural framework of the problem is represented by Young measures with finite second moment, instead of generalized Young measures considered in [**12**]; this fact sensibly simplifies the technical difficulties, making available, e.g., the tool of disintegration with respect to x (see Theorem 1.3.1). A Young measure approach has been proposed in [**32**], [**27**], [**37**], too. The main challenge in the Young measure formulation concerns the correlations between Young measures at subsequent time instants, which are involved in determining the total-dissipation energy of the system. In the mentioned papers, the definition of dissipation proposed by the authors seems to neglect some information: "Our approach will account only for dissipation losses if the distribution functions associated with the microstructure changes but not if the distributions stay fixed while the texture of the micro-pattern changes" ([**32**]).

In this thesis we follow the approach of [**12**], based on the notion of compatible systems, introduced in [**10**]. Roughly speaking, a compatible system is a family of Young measures indexed on all finite sequences of time instants; we require that if the measure associated to a time sequence $\{t_1, \dots, t_n\}$ is compared with the one corresponding to a subsequence $\{s_1 \dots, s_m\} \subset \{t_1, \dots, t_n\}$, a reasonable projection property is satisfied. The advantage of

[4] We refer to [**21**], [**29**], [**33**] for a complete proof.

[5] In Remark 3.4.3 we present an explicit example in which (1) has actually no solution.

this formulation consists in the capability of capturing the mutual interactions, occurring in passages to the limit, between oscillations at different time instants.

Since we do not need to use generalized Young measures, we are able to rephrase the definition of compatible system using a probabilistic language. In **Chapter 2**, an alternative formulation of compatible systems in terms of stochastic processes is provided using a modification of Kolmogorov's Theorem (see [**26**]): in Theorem 2.2.2, we prove that we can associate to any compatible system of Young measures with finite second moment a suitable stochastic process on a probability space of the form $(D \times \Omega, P)$. The variation accounting for the total-dissipation energy reduces in this language to the usual variation of the stochastic process $(\boldsymbol{Z}_t)_{t\in[0,T]}$, considered as a function from $[0,T]$ into $L^1(D \times \Omega; \mathbb{R}^m)$.

The probabilistic language is useful in order to better explain the necessity to deal with objects accounting simultaneously for more than one time instant: as the joint probability of two events can be expressed by the product of the probabilities only if the events are independent, in the same way the knowledge of the probability laws $\boldsymbol{Z}_s(P)$ and $\boldsymbol{Z}_t(P)$ is not always enough to determine the probability law $(\boldsymbol{Z}_s, \boldsymbol{Z}_t)(P)$.

Hence we deal with a generalized formulation, in which the quasistatic evolution is expressed in terms of stochastic processes $\boldsymbol{Z}_t \colon D\times\Omega \to \mathbb{R}^m$ and $\boldsymbol{Y}_t \colon D\times\Omega \to \mathbb{R}^{N\times d}$, for $t \in [0,T]$. In case stronger hypotheses allow to describe the evolution with functions $\boldsymbol{z}(t) \colon D \to \mathbb{R}^m$ and $\nabla\boldsymbol{v}(t) \colon D \to \mathbb{R}^{N\times d}$, they can be regarded as special stochastic processes $\boldsymbol{Z}_t$ and $\boldsymbol{Y}_t$ which do not depend on the variable ω. In the general case of $\boldsymbol{Z}_t$ and $\boldsymbol{Y}_t$ depending on both variables x and ω, the stochastic process represent the limit of oscillating functions, in a suitable topology; the role of the dependence on ω is to describe the statistics of these oscillations (see formula 2.2.2).

In **Chapter 3**, which contains the result of [**17**], the definition of a generalized notion of quasistatic evolution is given in terms of both Young measures and stochastic processes, and an existence result is proved, in a general framework ($Z = \mathbb{R}^m$).

The admissible set in which we look for the evolution is defined as the set of all stochastic processes $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ on $D\times\Omega$ with values in $\mathbb{R}^m\times\mathbb{R}^{N\times d}$, which can be approximated in a suitable way by means of functions on D satisfying the boundary condition. The proof of a closure property for the admissible set, needed to show the solvability of the incremental minimum problems, represents one of the most technical points of this chapter: it requires the usage of an equiintegrability result, in the version proposed by Fonseca, Müller, and Pedregal (*Decomposition Lemma*, see [**20**]), and a careful diagonalization argument.

A *globally stable* quasistatic evolution is defined as an admissible process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ satisfying the following conditions:

(ev1) *partial-global stability:* for every $t \in [0,T]$, we have

$$\int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega))\, dP(x,\omega) \leq \tag{2}$$
$$\leq \int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega) + \tilde{z}(x), \boldsymbol{Y}_t(x,\omega) + \nabla\tilde{u}(x))\, dP(x,\omega) + \int_D H(\tilde{z}(x))\, dx,$$

for every $\tilde{z} \in L^2(D;\mathbb{R}^m)$ and every $\tilde{u} \in H^1_0(D;\mathbb{R}^N)$;

(ev2) *energy inequality:* for every $t \in [0,T]$ we have

$$\int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega),\boldsymbol{Y}_t(x,\omega))\,dP(x,\omega)+\mathrm{Var}_H(\boldsymbol{Z},P;0,t)\leq$$
$$\leq\int_D W(z_0(x),\nabla v_0(x))\,dx+\int_0^t\langle\boldsymbol{\sigma}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds \tag{3}$$

where $\boldsymbol{\sigma}$ represents the stress (see Remark 3.3.10).

A basic tool in the proof of the existence theorem for this notion of evolution is a suitable version of Helly's selection principle for compatible systems of Young measures (see Theorem 2.1.6).

The stability condition we are able to prove for the evolution has a global character, since there is no restriction on the norm of the competitors in the minimality property (ev1); nevertheless, due to hard technical difficulties occurring in the passage to the limit of the approximate solutions, it is only possible to compare the evolution with its translation by functions on D, and this does not allow to choose arbitrary random variables $\tilde{Z}\colon D\times\Omega\to\mathbb{R}^m$, $\tilde{Y}\colon D\times\Omega\to\mathbb{R}^{N\times d}$ as competitors in (2). This is what we mean by "partial-global". For the same reason, only an energy inequality for time interval of the form $(0,t)$ can be obtained.

An interesting question is the comparison of this existence result with the one obtained in [**21**], i.e., can our result be regarded as a limit case of that analyzed by Francfort and Mielke with a spatial regularization? The answer is yes: under the more restrictive assumption that W is quasiconvex with respect to the deformation gradient, we show in Section 3.5 that the limit of the regularized evolutions converge to an admissible pair satisfying properties (ev1) and (ev2), as the regularization parameter tends to zero. This provides an easier proof of our existence result in this special case.

As noticed above, the notion of quasistatic evolution presented in **Chapter 3** is based on a global minimization procedure. One of the major drawbacks of this approach is the possibility for the evolution to perform abrupt jumps from one potential well to another one, without taking into account the barriers separating them. Therefore it would be preferable to follow a path composed by local minimizers rather than global minimizers.

As observed in [**32**], in this context it is nontrivial to find a suitable definition for the term "local"; moreover, the notion of locality should have some physical interpretation.

In the last years, many authors have dealt with this problem: see [**15**], [**11**], [**8**], [**45**], [**24**]. They study regularized problems which take into account the contribution of viscous forces, to obtain in the limit, as the regularization parameter vanishes, an evolution which does not jump over potential barriers.

In Chapter 4, which contains the results of [**18**], we follow the same approach, adapted to the framework of stochastic processes. The property of global stability is weakened and replaced by a

$(\mathrm{ev1})_{vv}$ *stationarity condition*: for every $t\in[0,T]$

$$-\mathrm{div}\boldsymbol{\sigma}(t)=0 \tag{4}$$
$$\boldsymbol{\zeta}(t)\in\partial\mathcal{H}(0) \tag{5}$$

where $\boldsymbol{\sigma}(t)$ is the stress, $\boldsymbol{\zeta}(t)$ is a dual variable associated to the stochastic process $\boldsymbol{Z}_t$ (see Remark 4.6.7 for the definition), and $\partial\mathcal{H}$ is the usual subdifferential of convex analysis.

Moreover, we want to adopt a selection criterion based on a sort of viscous approximation among stochastic processes satisfying (4), (5), and the energy inequality (3): we will accept as a solution of the quasistatic-evolution problem only those stochastic processes which are attainable as limit of solutions to suitable regularized evolution problems. For this reason this evolution will be called *approximable*.

These auxiliary problems can be solved in usual function spaces, and given a regularization parameter ε, they consist in finding a pair of functions $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$, satisfying the boundary condition and absolutely continuous with respect to time, such that the following equalities hold:

(a) *equilibrium condition:* for a.e. $t \in [0,T]$

$$-\mathrm{div}\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) = 0; \tag{6}$$

(b) *regularized flow rule:* for a.e. $t \in [0,T]$

$$\dot{\boldsymbol{z}}_\varepsilon(t) = N_K^\varepsilon(\boldsymbol{\zeta}_\varepsilon(t)) \quad \text{a.e. in } D,$$

where $N_K^\varepsilon(\zeta)$ is an approximation of the normal to K, defined by $N_K^\varepsilon(\zeta) := \frac{1}{\varepsilon}(\zeta - P_K(\zeta))$, P_K being the projection onto $K := \partial H(0)$.

An easy computation shows that conditions (a) and (b) for ε-regularized problems are equivalent to the following properties:

$(1)_\varepsilon$ *equilibrium condition:* for a.e. $t \in [0,T]$

$$-\mathrm{div}\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) = 0 \tag{7}$$

modified dual constraint: for a.e. $t \in [0,T]$

$$\boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t) \in \partial\mathcal{H}(0);$$

$(2)_\varepsilon$ *energy equality:* for every $t \in [0,T]$,

$$\begin{aligned}\mathcal{W}(\boldsymbol{z}_\varepsilon(t), \boldsymbol{v}_\varepsilon(t)) &+ \int_0^t \mathcal{H}(\dot{\boldsymbol{z}}_\varepsilon(s))\,ds + \varepsilon\int_0^t \|\dot{\boldsymbol{z}}_\varepsilon(s)\|_2^2\,ds + \\ &+ \varepsilon\int_0^t \langle\nabla\dot{\boldsymbol{v}}_\varepsilon(s), \nabla\dot{\boldsymbol{v}}_\varepsilon(s) - \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds = \mathcal{W}(z_0, v_0) + \\ &+ \int_0^t \langle\boldsymbol{\sigma}_\varepsilon(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds.\end{aligned}$$

Thanks to the effect of the viscous regularization, we can reproduce the standard arguments of time-discretization and construction of approximate solutions in the framework of Sobolev spaces, as in the convex case. Indeed, the incremental minimum problem corresponding to the evolution equations $(1)_\varepsilon$, $(2)_\varepsilon$ is as follows:

$$\min\ \left\{\mathcal{W}(z,v) + \mathcal{H}(z - z(t_{i-1})) + \tfrac{\varepsilon}{2\tau_i}\|z - z(t_{i-1})\|_2^2 + \tfrac{\varepsilon}{2\tau_i}\|\nabla v - \nabla v(t_{i-1})\|_2^2\right\}, \tag{8}$$

among all (z,v) admissible at time t_i, where τ_i is the time step $t_i - t_{i-1}$; under suitable regularity assumptions on W, the functional in (8) is strictly convex for τ_i sufficiently small, so that the minimizer exists and is unique.

The particular structure of these minimum problems allows to prove not only the existence in Sobolev spaces, but also the regularity with respect to time and the uniqueness of the solution $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ to the ε-regularized problem.

Unfortunately, in the limit as ε vanishes, $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ may develop stronger and stronger oscillations in space, since we loose the convexification effect of the regularizations and come back to the nonconvexity of the original problem. Therefore, in general we cannot expect to describe the approximable evolution in terms of functions rather than stochastic processes.

Theorem 4.6.8 proves that the limit as ε_k tends to 0 of a suitably chosen subsequence $(\boldsymbol{z}_{\varepsilon_k}, \boldsymbol{v}_{\varepsilon_k})$ satisfies the stationarity condition (4), (5), and the energy inequality (3), but also in this case we are not able to obtain neither a complete energy balance nor a uniqueness result.

The last part of Chapter 4 (Section 4.7) provides an explicit example which shows that the notion of globally stable quasistatic evolution and that one of approximable quasistatic evolution lead to different solutions. The simplified assumptions considered there allow to prove that there exists a unique approximable evolution, which lives in function spaces, and both the internal variable and the deformation gradient are spatially homogeneous. Theorem 4.7.2 shows that this evolution does not fulfil the requirements of the definition of globally stable evolution, since the stability condition is violated.

We have so far considered a very general setting with $Z = \mathbb{R}^m$. If we are interested in a material with a finite number of phases, it is reasonable to analyze the case in which Z is a finite set $\{\theta_\alpha \,:\, \alpha = 1, \dots, q\}$. In **Chapter 5**, which presents the results of [**19**], we consider the quasistatic evolution problem in this particular setting[6].

As in Chapter 3, the lack of convexity of the energy functional requires to study the problem in the extended framework of Young measures, but the fact that Z is finite makes it meaningless to consider translations of the internal variable or viscous regularizations. On the other hand the discrete setting brings significant advantages from a mathematical point of view; indeed, the Young measure which is going to substitute the internal variable is a measure on D with values in Z, hence it has finite moments of every order. We will see in the sequel that this point is crucial in order to improve the notion of quasistatic evolution with respect to the one considered in Chapter 3.

For this discrete case it seems to be more convenient to use the Young measure formulation instead of the probabilistic one, since the first one lends itself to a more evident mechanical interpretation.

To motivate this claim, let us recall that every probability measure on a finite set can be written as a linear combination of Dirac measures; in this sense the finite sequence of coefficients (whose sum is 1, since we deal with a probability) identifies the measure. In particular, this holds for the disintegration of any Young measure on D with values in Z.

In the same way, we can deduce that every compatible system on D with time set $[0, T]$ and values in Z can be represented by a suitable family of finite sequences of functions on D with values in $[0, 1]$, in the following way. Given a compatible system $\boldsymbol{\mu}$ and an index $\{t_1, \dots, t_n\}$, to the element $\boldsymbol{\mu}_{t_1 \dots t_n}$ is associated a sequence $(\boldsymbol{b}^{t_1 \dots t_n}_{\alpha_1 \dots \alpha_n})_{(\alpha_1, \dots, \alpha_n) \in \{1, \dots, q\}^n}$ in

[6] In this discrete setting we consider the more general case in which the dissipation rate H is a distance on Z, not necessarily generated by a norm.

$L^\infty(D;[0,1])$ through the relation:

$$\boldsymbol{\mu}^x_{t_1\dots t_n} = \sum_{(\alpha_1,\dots,\alpha_n)} \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n}(x)\delta_{(\theta_{\alpha_1},\dots,\theta_{\alpha_n})} \quad \text{for a.e. } x \in D,$$

where $\boldsymbol{\mu}^x_{t_1\dots t_n}$ represents the disintegration of $\boldsymbol{\mu}_{t_1\dots t_n}$, and $\delta_{(\theta_{\alpha_1},\dots,\theta_{\alpha_n})}$ is the Dirac measure centered in the vector $(\theta_{\alpha_1},\dots,\theta_{\alpha_n})$.

An analogous representation is proposed for the Young measure substituting the pair $(z,\nabla v)$ in the stored-energy functional, but in this case we have to deal with finite sequences both of coefficients and of Young measures, since the gradient of the deformation takes values in the continuous set $\mathbb{R}^{N\times d}$ (see Lemma 2.3.3).

In this language, when we deal with a Young measure on D with values in Z which is representable by a function z, the coefficient $b_\alpha(x)$ associated to δ_{θ_α} is given by

$$b_\alpha(x) = \begin{cases} 1 & \text{if } z(x) = \theta_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

In this particular case, the Young measure representation can be interpreted in the following way: the material assumes a *pure phase* distribution, i.e., to every point x is associated a pure phase $\theta \in Z$. While in the general case we say that the material has a *mixed phase* distribution meaning that at each point x we have a mixture of phases θ_α with volume fractions $b_\alpha(x)$.

The compatible systems describe the evolution of the phase distribution from a statistical point of view: if we consider two time instants $s < t$, $\boldsymbol{b}^{st}_{\alpha\beta}(x)$ represents the volume fraction at x undergoing the phase transition from θ_α at time s to θ_β at time t.

Thanks to these remarks, the formulation of a notion of quasistatic evolution in terms of a family of finite sequences of coefficients $\boldsymbol{b}$ and a family of finite sequences of Young measures $\boldsymbol{\lambda}$ seems to be the most appropriate one for the discrete case.

The set of admissible pairs $(\boldsymbol{b},\boldsymbol{\lambda})$ is described by approximation properties by functions, in this case taking values in $Z \times \mathbb{R}^{N\times d}$.

As in Chapter 3, we consider an evolution based on a global minimization process; indeed we cannot use the viscously regularized problems in the discrete case, since their solutions may not satisfy the constraint of taking values in $Z \times \mathbb{R}^N$.

The stability condition in terms of $(\boldsymbol{b},\boldsymbol{\lambda})$ is the following:

(ev1)$_d$ *partial-global stability:* for every $t \in [0,T]$ we have

$$\begin{aligned} &\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x) W(\theta_\alpha, F)\, d(\boldsymbol{\lambda}^t_\alpha)(x,F) \leq \\ &\leq \sum_{\alpha,\beta} \int_{D\times\mathbb{R}^{N\times d}} M_{\beta\alpha}(x)\boldsymbol{b}^t_\alpha(x) W(\theta_\beta, F + \nabla\tilde{u}(x))\, d\boldsymbol{\lambda}^t_\alpha(x,F) + \\ &\quad + \sum_{\alpha,\beta} H(\theta_\beta,\theta_\alpha) \int_D M_{\beta\alpha}(x)\boldsymbol{b}^t_\alpha(x)\, dx, \end{aligned} \tag{9}$$

for every $\tilde{u} \in H^1_0(D;\mathbb{R}^N)$, and every measurable function M on D with values in a special set of $q \times q$ real matrices.

The elements of this set are the matrices with nonnegative entries such that the sum of the entries of each column is 1; in probabilistic language they are called stochastic matrices (see, e.g., [**2**, Part 2]), and their entries $M_{\beta\alpha}$ represent the probability of a transition from phase θ_α to phase θ_β. In our model, $M_{\beta\alpha}(x)$ is the proportion of the volume fraction at x originally in phase θ_α undergoing a phase transition to θ_β. According to the picture described so far, the quantity

$$H(\theta_\beta, \theta_\alpha) M_{\beta\alpha}(x) \boldsymbol{b}^t_\alpha(x)$$

can be interpreted as the energy density dissipated at the point x by the phase transition from θ_α to θ_β. Therefore, the following expression

$$\sum_{\alpha,\beta} H(\theta_\beta, \theta_\alpha) \int_D M_{\beta\alpha}(x) \boldsymbol{b}^t_\alpha(x)\, dx$$

represents the energy which would be dissipated on the whole domain D, if we performed the microscopic phase transitions determined by M.

We observe that any other phase distribution $(\tilde{b}_\alpha)_\alpha$ can be obtained by the action of a suitable stochastic matrix: indeed, it is enough to choose $M_{\beta\alpha}(x) := \tilde{b}_\beta(x)$ for every α, β. Therefore, even though this notion of stability shares the partial character with the stability condition in Chapter 3, it seems to be better, since the minimality property is now satisfied with a quite large set of competitors including all possible rearrangements of the phase distribution.

From the stability property we can deduce a pointwise condition. If we call active at x the phases θ_α for which $\boldsymbol{b}^t_\alpha(x) > 0$, then the Euler equation for the internal variable can be written as follows: for a.e. x with active phase θ_α, we have

$$\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\, d(\boldsymbol{\lambda}^t_\alpha)^x(F) \leq \int_{\mathbb{R}^{N\times d}} W(\theta_\beta, F)\, d(\boldsymbol{\lambda}^t_\alpha)^x(F) + H(\theta_\beta, \theta_\alpha),$$

for every β (see Section 5.6). According to the above physical picture, this condition can be interpreted as a pointwise *optimality condition of the active phases*.

Clearly, an Euler equation for the deformation can be derived as well: it is the same classical equilibrium condition on the stress $\boldsymbol{\sigma}$ we have found in the case $Z = \mathbb{R}^m$ (see Remark 5.3.5 for the definition of $\boldsymbol{\sigma}$ in terms of $(\boldsymbol{b}, \boldsymbol{\lambda})$).

The energy inequality expressed in terms of $(\boldsymbol{b}, \boldsymbol{\lambda})$ takes the following form:

(ev2)$_d$ *energy inequality:* for every $t_1 \leq t_2$ in $[0, T]$ we have

$$\sum_{\alpha=1}^q \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^{t_2}_\alpha(x) W(\theta_\alpha, F)\, d\boldsymbol{\lambda}^{t_2}_\alpha(x, F) + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) \leq$$
$$\leq \sum_{\alpha=1}^q \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^{t_1}_\alpha(x) W(\theta_\alpha, F)\, d\boldsymbol{\lambda}^{t_1}_\alpha(x, F) + \int_{t_1}^{t_2} \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\, ds.$$

The dissipation $\mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2)$ is defined by

$$\mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) := \sup \sum_{i=1}^k \sum_{\alpha\beta} H(\theta_\beta, \theta_\alpha) \int_D \boldsymbol{b}^{s_{i-1}s_i}_{\alpha\beta}(x)\, dx, \tag{10}$$

where the supremum is taken over all partitions $t_1 = s_0 < \cdots < s_k = t_2$ of the interval $[t_1, t_2]$.

As remarked in [**41**], Young measures "determine the asymptotic local distribution of function values but contain no information about the direction, length scale or fine geometry of the oscillations", therefore it is still not clear which should be the right notion of dissipation energy given in terms of Young measures and solving the lack of information about micro-patterns highlighted in [**32**]. Nevertheless, in this discrete setting, it is possible to show how our definition of dissipation depending on functions $\boldsymbol{b}_{\alpha\beta}^{s_{i-1}s_i}$ (see (10)) fills in some gaps of the notions proposed in [**32**], [**27**], [**37**], which only take into account the contribution of single time instants ($\boldsymbol{b}_{\alpha}^{s_{i-1}}$ and $\boldsymbol{b}_{\beta}^{s_i}$ in our language). Indeed, consider the case of a homogeneous phase distribution $\boldsymbol{b}_{\alpha}^{s_{i-1}} = 1/q$ for every α; if we suppose that the material undergoes a transition from s_{i-1} to s_i just permuting the phases and leaving the volume fractions unchanged, we have $\boldsymbol{b}^{s_i} = \boldsymbol{b}^{s_{i-1}}$; hence the dissipation computed using only $\boldsymbol{b}^{s^{i-1}}$ and $\boldsymbol{b}^{s^i}$ is zero, while the dissipation energy computed using our formula with $\boldsymbol{b}^{s_{i-1}s_i}$ depends on the permutation and it is not zero.

Due to the partial character of the stability condition, we cannot still have a complete energy balance; nevertheless the energy inequality holds here for any pair of time instants, and not only for intervals of the form $(0, t)$.

In Theorem 5.4.2 we give an existence result for this improved notion of quasistatic evolution, assuming just suitable continuity and growth hypotheses on the energy density W and its partial derivatives with respect to the deformation gradient[7].

The proof of this theorem follows the classical scheme already mentioned; the new feature concerns the choice of the solutions to the discretized minimum problems. In the spirit of [**14**], we use the Ekeland Principle to select minimizers satisfying special approximability properties. Then the regularity results for quasi-minima of integral functionals (see [**3**] and [**22**]) are used to prove a uniform bound on the moments of order $2r > 2$ of the selected minimizers, and consequently of the approximate solutions. Clearly, the regularity argument applies only to the "gradient part" of the Young measure substituting $(z, \nabla v)$, i.e., to the projection over the set $D \times \mathbb{R}^{N\times d}$ of this measure; this is the reason why it is fundamental that the "internal variable part" of the measure, i.e., the projection over $D \times Z$, has equibounded moments of every order.

As a by-product of this selection procedure in the construction of approximate solutions $(\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t)$, we get the continuity of the functional

$$(\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t) \mapsto \int_{D\times\mathbb{R}^{N\times d}} \sum_{\alpha} (\boldsymbol{b}_n^t)_\alpha(x) W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_n^t)_\alpha(x, F).$$

Thanks to this continuity property, we are able to obtain in the limit the stability condition and the energy inequality written above. A technical difficulty in the proof of the stability condition is the approximation of the right hand-side of (9) by integrals corresponding to functions satisfying the prescribed boundary condition. This is done by adapting to our problem the classical Riemann Lebesgue Lemma.

[7] These assumptions are actually weaker than those in Chapter 3 and Chapter 4, where it is necessary to require some condition of Lipschitz type on W.

CHAPTER 1

Mathematical preliminaries

In this chapter we fix some notation and recall some results which will be useful in the sequel.

In the whole thesis D will be a bounded, connected, open subset of $\mathbb{R}^d$, for $d \geq 1$, while Ξ will denote a finite dimensional Hilbert space.

1.1. Functions

The Lebesgue measure on $\mathbb{R}^d$ is usually denoted by $\mathcal{L}^d$, while $\mathcal{H}^k$ is the k-dimensional Hausdorff measure; we sometimes use the notation $|E|$ for the Lebesgue measure of a measurable subset E of $\mathbb{R}^d$. The Borel σ-algebra on D is denoted by $\mathcal{B}(D)$. The symbol 1_B indicates the characteristic function of a subset B of $\mathbb{R}^d$.

For $1 \leq p \leq +\infty$, $\|\cdot\|_p$ is the usual norm on L^p, while $W^{1,p}(D;\mathbb{R}^N)$ denotes the usual Sobolev space of all L^p functions from an open domain $D \subseteq \mathbb{R}^d$ into $\mathbb{R}^N$, with L^p first derivatives. We indicate $W^{1,2}(D;\mathbb{R}^N)$ with $H^1(D;\mathbb{R}^N)$. The symbol $\langle\cdot,\cdot\rangle$ will denote a duality pairing depending on the context.

Given a function $f \in L^1(D)$ and a measurable subset $E \subseteq D$, the mean value of f over E is denoted by $(f)_E$, i.e.

$$(f)_E := \tfrac{1}{|E|}\int_E f(x)\,dx = ⨍_E f(x)\,dx.$$

We recall the well-known following lemma.

Lemma 1.1.1. *Let $f \in L^2(D)$, and consider a finite measurable partition $(D_i)_{i=1}^I$ of D. The projection of f onto the space*

$$\mathcal{K} := \{g \in L^2(D) \ : \ g_{|D_i} \text{ is constant for every } i = 1, \dots, I\}$$

is

$$P_{\mathcal{K}}(f) := \sum_{i=1}^I (f)_{D_i} 1_{D_i}.$$

The classical Riemann Lebesgue Lemma (see, e.g., [**43**, pg. 121]) will be fundamental for the results in the last chapter.

Lemma 1.1.2 (Riemann Lebesgue Lemma). *Let us consider $f \in L^\infty([0,1]^d;\mathbb{R}^m)$ and extend it by periodicity to all $\mathbb{R}^d$. The sequence $f_\delta(x) := f(\frac{x}{\delta})$ satisfies*

$$f_\delta \rightharpoonup \int_{[0,1]^d} f(x)\,dx \quad L^\infty\text{-weakly*},$$

as $\delta \to 0$.

A function $f\colon D \times \Xi \to \mathbb{R}$ is said to be a *Carathéodory function* if $f(x,\cdot)$ is continuous for a.e. $x \in D$ and $f(\cdot,\xi)$ is measurable for every $\xi \in \Xi$. The space of all continuous functions $\phi\colon D \times \Xi \to \mathbb{R}$, such that $|\phi| \geq \varepsilon$ is compact for every $\varepsilon > 0$, is denoted by $\mathcal{C}_0(D \times \Xi)$. The following classical result (see, e.g., [**16**, Section 1.3]) will be very useful in the next chapter.

LEMMA 1.1.3 (Scorza-Dragoni Lemma). *Let A and B be Borel subsets of $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively, with A compact. Consider a Carathéodory function $f\colon D \times B \to \mathbb{R}$. Then for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset A$ such that $|A \setminus K_\varepsilon| \leq \varepsilon$ for which the restriction of f to $K_\varepsilon \times B$ is continuous.*

The symbol $\mathbf{M}^{q\times q}_{St}$ denotes the set of all stochastic matrices of size $q \times q$, i.e. the set of all matrices $(M_{\beta\alpha})_{\beta\alpha}$ with

- $0 \leq M_{\beta\alpha} \leq 1$ for every α, β,
- $\sum_\beta M_{\beta\alpha} = 1$, for every α.

Before concluding this section, we want to to point out some properties and to introduce some new notations related to the integral functional $\mathcal{H}\colon L^1(D;\mathbb{R}^m) \to \mathbb{R}$, defined by

$$\mathcal{H}(z) := \int_D H(z(x))\, dx,$$

where $H\colon \mathbb{R}^m \to [0,+\infty)$ satisfies the following assumptions:

- H is positively homogeneous of degree one and convex;
- there exists a positive constant λ, such that $\frac{1}{\lambda}|\theta| \leq H(\theta) \leq \lambda|\theta|$.

From the hypotheses on H, it follows that $\mathcal{H}$ is lower semicontinuous with respect to the weak topology of $L^2(D;\mathbb{R}^m)$ and satisfies the triangle inequality, i.e.

$$\mathcal{H}(z_1 + z_2) \leq \mathcal{H}(z_1) + \mathcal{H}(z_2).$$

For every $\varepsilon > 0$ we define the function $H_\varepsilon\colon \mathbb{R}^m \to \mathbb{R}$ as

$$H_\varepsilon(\theta) := H(\theta) + \frac{\varepsilon}{2}|\theta|^2, \tag{1.1.1}$$

and the corresponding integral functional $\mathcal{H}_\varepsilon\colon L^2(D;\mathbb{R}^m) \to \mathbb{R}$ as

$$\mathcal{H}_\varepsilon(z) := \int_D H_\varepsilon(z(x))\, dx. \tag{1.1.2}$$

The convex conjugate $H^*_\varepsilon\colon \mathbb{R}^m \to \mathbb{R}$ of H_ε is

$$H^*_\varepsilon(\zeta) := \sup_{\theta\in\mathbb{R}^m} \{\zeta\theta - H_\varepsilon(\theta)\}.$$

Since the convex conjugate H^* of H is the indicator function of the convex set $K := \partial H(0)$ (see [**44**, Theorem 13.2]), using [**44**, Theorem 16.4], it can be proved that

$$H^*_\varepsilon(\zeta) = \tfrac{1}{2\varepsilon}|\zeta - P_K(\zeta)|^2, \tag{1.1.3}$$

where $P_K\colon \mathbb{R}^m \to K$ is the projection onto K. Therefore H^*_ε is differentiable with gradient

$$N^\varepsilon_K(\zeta) := \tfrac{1}{\varepsilon}(\zeta - P_K(\zeta)). \tag{1.1.4}$$

In particular N^ε_K is Lipschitz continuous.

Let $\mathcal{H}_\varepsilon^*\colon L^2(D;\mathbb{R}^m) \to \mathbb{R}$ be the convex conjugate of $\mathcal{H}_\varepsilon$. It can be easily shown (using a general property of integral functionals, see e.g., [**16**, Proposition IX.2.1]) that

$$\mathcal{H}_\varepsilon^*(\zeta) = \int_D H_\varepsilon^*(\zeta(x))\ dx,$$

so that the gradient $\partial\mathcal{H}_\varepsilon^*$ is given by

$$\partial\mathcal{H}_\varepsilon^*(\zeta)(x) = N_K^\varepsilon(\zeta(x)), \quad \text{for a.e. } x \in D. \tag{1.1.5}$$

Therefore $\partial\mathcal{H}_\varepsilon^*$ is Lipschitz continuous.

1.2. Quasi-minima

The results in the last chapter are based on some regularity theorems concerning quasi-minima of integral functionals. We now briefly recall the notion of cubic quasi-minimum, introduced by Giaquinta and Giusti in [**22**], and the related results.

Given $\varphi \in H^1(D;\mathbb{R}^N)$, let $\mathcal{G}$ be the functional defined by

$$\mathcal{G}(v) = \mathcal{G}(v, D) := \int_D G(x, \nabla v(x))\ dx$$

for every $v \in \varphi + H_0^1(D;\mathbb{R}^N)$, where $G\colon D \times \mathbb{R}^{N\times d} \to \mathbb{R}$ is a function satisfying

$$|G(x,F)| \le L(|F|^2 + 1)$$

for a suitable positive constant L, for every $(x,F) \in D \times \mathbb{R}^{N\times d}$.

Definition 1.2.1. Let $Q > 0$. A function $v \in H^1(D;\mathbb{R}^{N\times d})$ is said to be a *cubic Q-quasi-minimum for the functional $\mathcal{G}$ if for every cube of side R, $Q_R \subset\subset D$, and for every function $w \in H^1(D;\mathbb{R}^{N\times d})$, with* $\operatorname{supp}(v-w) \subseteq Q_R$, *we have*

$$\mathcal{G}(v, Q_R) \le Q\mathcal{G}(w, Q_R).$$

We restrict our analysis to the particular case of $G(F) = 1 + |F|^2$, since this is the integrand we will consider; for the reader's convenience, we recall the statement and the proof of the Caccioppoli inequality for quasi-minima of the corresponding integral functional: for our purposes, we need a slightly different statement of the result contained in [**23**, Theorem 6.5]; our statement does not involve the L^{2^*}-norm of the quasi-minimum but it is valid for every cube Q_R. The precise result we will use is the following.

Theorem 1.2.2. *Let $v \in H^1(D;\mathbb{R}^N)$ be a Q-cubic quasi-minimum of the functional*

$$\mathcal{G}(w) = \int_D (1 + |\nabla w|^2)\ dx.$$

Then there exist a positive constant $C > 0$, depending only on Q, such that

$$\fint_{Q_{R/2}} |\nabla v|^2\ dx \le C\Big\{\Big(\fint_{Q_R} |\nabla v|^{2m}\ dx\Big)^{\frac{1}{m}} + 1\Big\}, \tag{1.2.1}$$

for every cube $Q_R \subset\subset D$, where $m = \dfrac{d}{2+d}$.

Proof. Let $R/2 < t < s \le R$. We consider a cut-off function $\eta \in \mathcal{C}_0^\infty(Q_s)$, with $0 \le \eta \le 1$, $\eta \equiv 1$ on Q_t, and $|\nabla\eta| \le \frac{2}{s-t}$. Let $\phi := \eta(v - v_s)$, where v_s denotes the mean value $(v)_{Q_s}$ of v over Q_s; define a function w by $w := v - \phi$, so that $w = v_s + (1-\eta)(v - v_s)$. We have

$$\int_{Q_s} |\nabla\phi|^2 \, dx \le \int_{Q_s} (|\nabla v|^2 + 1) \, dx + \int_{Q_s} \big||\nabla\phi|^2 - |\nabla v|^2\big| \, dx; \tag{1.2.2}$$

Since by construction $\nabla\phi = \nabla v$ on Q_t, we have

$$\begin{aligned}\int_{Q_s} \big||\nabla\phi|^2 - |\nabla v|^2\big| \, dx &= \int_{Q_s\setminus Q_t} \big||\nabla\phi|^2 - |\nabla v|^2\big| \, dx \le \\ &\le 2\Big[\int_{Q_s\setminus Q_t} |\nabla v|^2 \, dx + \int_{Q_s} |\nabla w|^2 \, dx\Big].\end{aligned} \tag{1.2.3}$$

Moreover, by the quasi-minimum property of v, we have

$$\int_{Q_s} (|\nabla v|^2 + 1) \, dx \le Q \int_{Q_s} (|\nabla w|^2 + 1) \, dx; \tag{1.2.4}$$

therefore (1.2.2), (1.2.3), and (1.2.4) imply

$$\begin{aligned}\int_{Q_t} |\nabla v|^2 \, dx &= \int_{Q_t} |\nabla\phi|^2 \, dx \le \int_{Q_s} |\nabla\phi|^2 \, dx \le \\ &\le (Q+2)\int_{Q_s} (|\nabla w|^2 + 1) \, dx + 2\int_{Q_s\setminus Q_t} |\nabla v|^2 \, dx.\end{aligned} \tag{1.2.5}$$

Using the relation

$$|\nabla w|^2 = |(1-\eta)\nabla v + (v - v_s)\nabla\eta|^2 \le c[(1-\eta)^2|\nabla v|^2 + (s-t)^{-2}|v - v_s|^2],$$

we obtain from (1.2.5)

$$\begin{aligned}\int_{Q_t} |\nabla v|^2 \, dx &\le (c+1)(Q+2)\Big\{\int_{Q_s\setminus Q_t} |\nabla v|^2 \, dx + \\ &+ \frac{1}{(s-t)^2}\int_{Q_s} |v - v_s|^2 \, dx + |Q_s|\Big\}.\end{aligned} \tag{1.2.6}$$

Since we have

$$\int_{Q_s} |v - v_s|^2 \, dx \le c\int_{Q_R} |v - v_R|^2 \, dx,$$

(1.2.6) implies

$$\begin{aligned}\int_{Q_t} |\nabla v|^2 \, dx &\le (c+1)(Q+2)\Big\{\int_{Q_s\setminus Q_t} |\nabla v|^2 \, dx + \\ &+ \frac{1}{(s-t)^2}\int_{Q_R} |v - v_R|^2 \, dx + |Q_s|\Big\}.\end{aligned} \tag{1.2.7}$$

Now we use the "hole filling" method: we add to both terms of (1.2.7) the quantity

$$(c+1)(Q+2)\int_{Q_t} |\nabla v|^2 \, dx,$$

to get

$$\int_{Q_t} |\nabla v|^2 \, dx \le \theta \int_{Q_s \setminus Q_t} |\nabla v|^2 \, dx + \frac{1}{(s-t)^2} \int_{Q_R} |v - v_R|^2 \, dx + |Q_R|, \qquad (1.2.8)$$

with $1 > \theta := \frac{(c+1)(Q+2)}{(c+1)(Q+2)+1}$. Therefore, we are in the position to apply the same technical Lemma as in [**23**] (see [**23**, Lemma 6.1]), obtaining

$$\int_{Q_{R/2}} |\nabla v|^2 \, dx \le c\Big\{ \frac{1}{R^2} \int_{Q_R} |v - v_R|^2 \, dx + |Q_R| \Big\}. \qquad (1.2.9)$$

Set now $2_* := \frac{2d}{2+d}$, we have $2_* < d$ and

$$(2_*)^* = \frac{2_* d}{d - 2_*} = 2;$$

hence, by the Sobolev-Poincaré inequality (see [**23**, formula (3.32)]), we have

$$\int_{Q_R} |v - v_R|^2 \, dx \le c\Big(\int_{Q_R} |\nabla v|^{2_*} \, dx \Big)^{2/2_*} = c\Big(\int_{Q_R} |\nabla v|^{2m} \, dx \Big)^{1/m},$$

which together with (1.2.9) gives (1.2.1). □

If we deal with quasi-minima satifying a prescribed boundary condition, the following result can be proved with similar arguments (see [**23**, Section 6.5]).

Theorem 1.2.3. *Let $V \in W^{1,p}(D;\mathbb{R}^N)$, for $2 < p$, and let $v \in V + H^1_0(D;\mathbb{R}^N)$ be a Q-cubic quasi-minimum of the functional*

$$\mathcal{G}(w) = \int_D (1 + |\nabla w(x)|^2) \, dx,$$

i.e., for every cube $Q_R \subset \mathbb{R}^d$, and every $w \in H^1(D;\mathbb{R}^N)$ such that $v - w \in H^1_0(D \cap Q_R)$ we have

$$\int_{(Q_R \cap D)} (1 + |\nabla v|^2) \, dx \le Q \int_{Q_R \cap D} (1 + |\nabla w|^2) \, dx.$$

Then there exist a positive constant $C > 0$, depending only on Q, such that

$$⨍_{Q_{R/2}} |\nabla(v - V)|^2 \, dx \le C\Big\{ \Big(⨍_{Q_R} |\nabla(v - V)|^{2m} \, dx \Big)^{\frac{1}{m}} + 1 \Big\}, \qquad (1.2.10)$$

for every cube $Q_R \subset \mathbb{R}^d$, where $m = \dfrac{d}{2+d}$ and $v - V$ is extended to 0 in $Q_R \setminus D$.

Using Theorem 1.2.2 and Theorem 1.2.3, we can obtain as in [**23**, Theorem 6.8] the following result

Theorem 1.2.4. *Let $V \in W^{1,p}(D;\mathbb{R}^N)$, for $2 < p$, and let $v \in V + H^1_0(D;\mathbb{R}^N)$ a Q-cubic quasi-minimum of the functional*

$$\mathcal{G}(w) = \int_D (1 + |\nabla w(x)|^2) \, dx.$$

Then there exist constants $\gamma > 0$ and $r > 1$, depending only on Q and V, such that

$$\int_D |\nabla v|^{2r} \, dx \le \gamma\Big\{ \big(\int_D |\nabla v|^2 \, dx \big)^r + 1 \Big\}. \qquad (1.2.11)$$

1.3. Measures

In this section we collect some technical tools concerning measures in general, and Young measures in particular.

We will use the following notation: π_D and π_Ξ will denote the usual projections of the product space $D\times\Xi$ on D and Ξ respectively; in the case $\Xi=\Xi_1\times\Xi_2$, $\tilde{\pi}_{\Xi_i}$ will denote the projection of $D\times\Xi_1\times\Xi_2$ on $D\times\Xi_i$ and π_{Ξ_i} the projection of $\Xi_1\times\Xi_2$ on Ξ_i, for $i=1,2$.

We denote by $M_b(D\times\Xi)$ the space of all bounded Radon measures on $D\times\Xi$; this space can be identified with the dual of the Banach space $\mathcal{C}_0(D\times\Xi)$. We will consider on $M_b(D\times\Xi)$ the weak* topology deriving from this duality.

Let $(A,\mathcal{F})$ be a measure space, Ξ a finite dimensional Hilbert space, and $\mu\in M_b(D\times\Xi)$; for every $\mathcal{B}(D)$-$\mathcal{F}$-measurable function $f\colon D\times\Xi\to A$, the *image measure*, defined by $\mu(f^{-1}(B))$ for every measurable set $B\subseteq A$, will be denoted by $f(\mu)$; for every bounded measurable function $g\colon D\to\mathbb{R}$, the product $g\mu$ is defined by

$$\int_{D\times\Xi}\phi(x,\xi)\ d(g\mu)(x,\xi):=\int_{D\times\Xi}g(x)\phi(x,\xi)\ d\mu(x,\xi),$$

for every bounded Borel function $\phi\colon D\times\Xi\to\mathbb{R}$.

We recall the following classical result (see, e.g., [**46**, Appendix A2]).

THEOREM 1.3.1 (Disintegration Theorem). *Let ν and μ be nonnegative measures in $M_b(D)$ and $M_b(D\times\Xi)$, respectively, such that $\pi_D(\mu)=\nu$. Then there exists a measurable family $(\mu^x)_{x\in D}$ of probability measures on Ξ, such that*

$$\int_{D\times\Xi}f(x,\xi)d\mu(x,\xi)=\int_D\Big(\int_\Xi f(x,\xi)d\mu^x(\xi)\Big)d\nu(x),$$

for every bounded Borel function $f\colon D\times\Xi\to\mathbb{R}$. The measures μ^x are uniquely determined for a.e. $x\in D$ and we will write

$$\mu=\int_D\mu^x\ d\nu(x). \tag{1.3.1}$$

The space $Y(D;\Xi)$ of the *Young measures* on D with values in Ξ is the space of all nonnegative measures $\mu\in M_b(D\times\Xi)$ such that

$$\pi_D(\mu)=\mathcal{L}^d. \tag{1.3.2}$$

Applying the Disintegration Theorem to $\mu\in Y(D;\Xi)$, we deduce the existence of a measurable family of probability measures on Ξ, $(\mu^x)_{x\in D}$, with

$$\mu=\int_D\mu^x\ dx.$$

Fixed $p\geq 1$, $Y^p(D;\Xi)$ denotes the space of all $\mu\in Y(D;\Xi)$, whose *p-moment*

$$\int_{D\times\Xi}|\xi|^p d\mu(x,\xi)=\int_D\Big(\int_\Xi|\xi|^p\ d\mu^x(\xi)\Big)\ dx$$

is finite.

Given $\xi_0\in\Xi$, the measure $\delta_{\xi_0}\in M_b(\Xi)$ is defined by

$$\int_\Xi f(\xi)\ d\delta_{\xi_0}(\xi)=f(\xi_0),$$

for every bounded Borel function $f\colon \Xi \to \mathbb{R}$; fixed a $\mathcal{B}(D)$-$\mathcal{B}(\Xi)$-measurable function $u\colon D \to \Xi$, the Young measure $\boldsymbol{\delta}_u \in Y(D;\Xi)$ is defined by

$$\int_{D\times\Xi} g(x,\xi)\ d\boldsymbol{\delta}_u(x,\xi) = \int_D g(x,u(x))\ dx,$$

for every bounded Borel function $g\colon D\times\Xi \to \mathbb{R}$. In particular $\boldsymbol{\delta}_{\xi_0}$ is the Young measure associated to the constant function $u(x) \equiv \xi_0$, which should not be confused with the measure δ_{ξ_0}.

We say that a sequence μ_k in $Y(D;\Xi)$ *weakly** converges if it converges in the weak* topology of $M_b(D\times\Xi)$.

REMARK 1.3.2. Since the total variation of a Young measure μ is $|\mu|(D\times\Xi) = \mathcal{L}^d(D)$, $Y(D;\Xi)$ is contained in a bounded subset of the dual of a separable Banach space, therefore it is metrizable with respect to the weak* topology.

In this thesis we deal with Young measures with finite p-moment, therefore it is worth defining the following notion of convergence.

DEFINITION 1.3.3. We will say that $\mu_k \rightharpoonup \mu$ *p-weakly** if the p-moments of μ_k are equibounded and $\mu_k \rightharpoonup \mu$ weakly*.

We recall that $Y(D;\Xi)$ is not closed with respect to the weak* convergence, differently from $Y^p(D;\Xi)$, for $p>1$, which is closed under p-weakly* convergence, as we can deduce from the following remark.

REMARK 1.3.4. If μ_k is a sequence in $Y(D;\Xi)$ and μ_k weakly* converges to some $\mu \in M_b(D\times\Xi)$, then for every positive Carathéodory function f on $D\times\Xi$ we have

$$\int_{D\times\Xi} f(x,\xi)\ d\mu(x,\xi) \leq \liminf_{k\to\infty} \int_{D\times\Xi} f(x,\xi)\ d\mu_k(x,\xi), \tag{1.3.3}$$

(see [**46**, Theorem 4]). In particular, if a sequence $(\mu_k)_k \subset Y^p(D;\Xi)$ p-weakly* converges to a measure $\mu \in M_b(D\times\Xi)$, then $\mu \in Y^p(D;\Xi)$; moreover from the previous remark we can deduce that a sequence in $Y^p(D;\Xi)$ with equibounded p-moments has always a subsequence which converges p-weakly*.

The following lemma is a slight modification of [**43**, Proposition 6.5, pg.103].

LEMMA 1.3.5. *Let $1 < p \leq \infty$, and let $(\mu_k)_k \subseteq Y^p(D;\Xi)$ satisfy $\mu_k \rightharpoonup \mu$ p-weakly*, for a suitable $\mu \in Y^p(D;\Xi)$. Then, for every Carathéodory function $f\colon D\times\Xi \to \mathbb{R}$, with $|f(x,\xi)| \leq a(x) + b(x)|\xi|^q$, for every $x \in D$, $\xi \in \Xi$, for suitable $1 \leq q < p$, b a nonnegative function in $L^{\frac{p}{p-q}}(D)$, and a a nonnegative function in $L^1(D)$, it holds*

$$\int_{D\times\Xi} f(x,\xi)\,d\mu_k(x,\xi) \longrightarrow \int_{D\times\Xi} f(x,\xi)\,d\mu(x,\xi).$$

In particular, if $\mu_k \rightharpoonup \mu$ p-weakly, then $\tilde{\pi}_i(\mu_k) \rightharpoonup \tilde{\pi}_i(\mu)$ in $Y^p(D;\Xi_i)$, as $k\to\infty$, for $i = 1,2$.*

PROOF. It is enough to prove the statement for $f \geq 0$.

Let us suppose first that f has compact support and is uniformly bounded by a constant N. By Lemma 1.1.3 , for every $\varepsilon > 0$, there exists a measurable subset D_ε of D, such that $|D_\varepsilon| < \varepsilon$, $D \setminus D_\varepsilon$ is compact, and the restriction to $D \setminus D_\varepsilon$ of f is continuous. Hence $f \in \mathcal{C}_0((D \setminus D_\varepsilon) \times \Xi)$. Therefore, for every $\varepsilon > 0$ we have

$$\int_{(D\setminus D_\varepsilon)\times\Xi} f(x,\xi)\ d\mu_k(x,\xi) \to \int_{(D\setminus D_\varepsilon)\times\Xi} f(x,\xi)\ d\mu(x,\xi),$$

as $k \to \infty$. On the other hand, we have

$$\sup_k \int_{D_\varepsilon\times\Xi} f(x,\xi)\ d\mu_k(x,\xi) \le \sup_k N\mu_k(D_\varepsilon \times \Xi) = N|D_\varepsilon| \le N\varepsilon \to 0,$$

as $\varepsilon \to 0$, and the same holds for μ. Hence in this case the result is proved.

Let us consider now the case of f uniformly bounded by a positive constant N, and such that $\xi \mapsto f(x,\xi)$ has compact support $K \subset \Xi$, for a.e. $x \in D$. For every $\delta > 0$, let D_δ be a measurable subset of D with $|D_\delta| \le \delta$, and consider a cut-off function $\eta_\delta \colon D \to [0,1]$, with $\eta_{\delta|D\setminus D_\delta} \equiv 1$. The function $\eta_\delta f$ is uniformly bounded with compact support; hence, by the previous step of the proof, we know that for every $\delta > 0$

$$\int_{D\times\Xi} \eta_\delta(x) f(x,\xi)\ d\mu_k(x,\xi) \to \int_{D\times\Xi} \eta_\delta(x) f(x,\xi)\ d\mu(x,\xi),$$

as $k \to \infty$. On the other hand, we have

$$\sup_k \int_{D\times\Xi} (1-\eta_\delta(x)) f(x,\xi)\ d\mu_k(x,\xi) \le \sup_k N\mu_k(D_\delta \times \Xi) = N|D_\delta| \le N\delta \to 0,$$

as $\delta \to 0$, and the same holds for μ. Therefore, also in this case we have proved the thesis.

Suppose now that $\xi \mapsto f(x,\xi)$ has compact support $K \subset \Xi$, for a.e. $x \in D$, and that there exists a function $c \in L^1(D;\mathbb{R})$ satisfying $f(x,\xi) \le c(x)$ for every ξ and a.e. x. Let $f_N \colon D \times \Xi \to [0,\infty)$ be defined by

$$f_N(x,\xi) := \min\{f(x,\xi), \min\{N, c(x)\}\},$$

for every ξ and a.e. x, and for every $N > 0$. It is immediate to see that

$$f_N(x,\xi) = \begin{cases} N & \text{if } N \le c(x) \text{ and } N \le f(x,\xi) \\ f(x,\xi) & \text{otherwise} \end{cases}.$$

In particular $f_N(x,\xi) \le N$, for every ξ and for a.e. x. Therefore, by the previous step, for every $N > 0$ we have

$$\int_{D\times\Xi} f_N(x,\xi)\ d\mu_k(x,\xi) \to \int_{D\times\Xi} f_N(x,\xi)\ d\mu(x,\xi),$$

as $k \to \infty$. On the other hand, $|f(x,\xi) - f_N(x,\xi)| \le (c(x) - N)^+$, so that

$$\sup_k \int_{D\times\Xi} |f(x,\xi) - f_N(x,\xi)|\ d\mu_k(x,\xi) \le \|(c(\cdot) - N)^+\|_1 \to 0,$$

as $N \to \infty$, and the same holds for μ. Hence we can conclude that the result is true also in this case.

Finally, let us consider the general case. For every $t \in \mathbb{R}$, define a cut-off function $\eta_t \in \mathcal{C}^\infty(\Xi,[0,1])$, with $\operatorname{supp}(\eta_t) \subseteq \{|\xi| < t+1\}$ and $\eta_{t|\{|\xi|\le t\}} \equiv 1$. Thanks to the growth hypothesis on f, $\eta_t(\xi)f(x,\xi) \le a(x) + b(x)(t+1)^q$, for every t; therefore we are in the hyptheses of the previous case and

$$\int_{D\times\Xi} \eta_t(\xi)f(x,\xi)\ d\mu_k(x,\xi) \to \int_{D\times\Xi} \eta_t(\xi)f(x,\xi)\ d\mu(x,\xi)$$

as $k \to \infty$. To conclude, it is now enough to show that

$$\sup_k \Big| \int_{D\times\Xi} (1-\eta_t(\xi))f(x,\xi)\ d\mu_k(x,\xi) \Big| \to 0, \tag{1.3.4}$$
$$\Big| \int_{D\times\Xi} (1-\eta_t(\xi))f(x,\xi)\ d\mu(x,\xi) \Big| \to 0,$$

as $t \to \infty$.

To this end, let us suppose first that b is a constant function, and consider for every n the function

$$f_n(x,\xi) := \min\{f(x,\xi), \min\{a(x),n\} + b|\xi|^q\},$$

for a.e. $x \in D$ and every $\xi \in \Xi$. It is immediate to see that

$$f_n(x,\xi) = \begin{cases} n + b|\xi|^q & \text{if } n \le a(x) \text{ and } n + b|\xi|^q \le f(x,\xi) \\ f(x,\xi) & \text{otherwise.} \end{cases}$$

Hence $|f(x,\xi) - f_n(x,\xi)| \le (a(x)-n)^+$, so that

$$\sup_k \int_{D\times\Xi} |f(x,\xi) - f_n(x,\xi)|\ d\mu_k(x,\xi) \le \|(a(\cdot)-n)^+\|_1 \to 0,$$

as $n \to \infty$. On the other hand $0 \le f_n(x,\xi) \le n + b|\xi|^q$; therefore, we have

$$\begin{aligned} 0 \le \int_{D\times\Xi} (1-\eta_t(\xi))f_n(x,\xi)\ d\mu_k(x,\xi) &\le \int_{\{(x,\xi)\,:\,|\xi|>t\}} f_n(x,\xi)\ d\mu_k(x,\xi) \le \\ &\le \frac{1}{t^{p-q}} \int_{D\times\Xi} |\xi|^{p-q}(n + b|\xi|^q)\ d\mu_k(x,\xi) \le \\ &\le \frac{1}{t^{p-q}} [n|D|^{q/p} + b]\Big(1 + \int_{D\times\Xi} |\xi|^p\ d\mu_k(x,\xi)\Big); \end{aligned} \tag{1.3.5}$$

since the p-moments of μ_k are uniformly bounded, we deduce from (1.3.5) that

$$\sup_k \int_{D\times\Xi}\int_{D\times\Xi} (1-\eta_t(\xi))f_n(x,\xi)\ d\mu_k(x,\xi) \to 0,$$

as $t \to \infty$, and the same occurs to μ. Hence we have shown that (1.3.4) holds for every positive function bounded from above by $a(x) + b|\xi|^q$.

Consider now the general case of $b \in L^{\frac{p}{p-q}}(D)$. For every m define

$$f_m(x,\xi) := \min\{f(x,\xi), a(x) + \min\{b(x),m\}|\xi|^q\}$$

for a.e. $x \in D$ and every $\xi \in \Xi$. As before, we have

$$f_m(x,\xi) = \begin{cases} a(x) + m|\xi|^q & \text{if } m \le b(x) \text{ and } a(x) + m|\xi|^q \le f(x,\xi) \\ f(x,\xi) & \text{otherwise,} \end{cases}$$

so that $|f(x,\xi) - f_m(x,\xi)| \le (b(x) - m)^+ |\xi|^q$. Therefore

$$\sup_k \int_{D\times\Xi} |f(x,\xi) - f_m(x,\xi)|\ d\mu_k(x,\xi) \le \|(b(\cdot) - m)^+\|_{\frac{p}{p-q}} \sup_k \Big(\int_{D\times\Xi} |\xi|^p\ d\mu_k(x,\xi) + 1\Big),$$

which vanishes as $m \to \infty$, thanks to the uniform bound on the p-moment of μ_k. On the other hand $0 \le f_m(x,\xi) \le a(x) + m|\xi|^q$, so that we can apply the partial result obtained above to deduce that, fixed m,

$$\sup_k \int_{D\times\Xi} (1 - \eta_t(\xi)) f_m(x,\xi)\ d\mu_k(x,\xi) \to 0,$$
$$\int_{D\times\Xi} (1 - \eta_t(\xi)) f_m(x,\xi)\ d\mu(x,\xi) \to 0,$$

as $t \to \infty$, which concludes the proof. $\square$

For every $g \in L^p(D;\Xi)$, the translation map $\mathcal{T}_g$ from $D \times \Xi$ into itself is defined by $\mathcal{T}_g(x,\xi) := (x, \xi + g(x))$. We can deduce from Lemma (1.3.5) the following result.

LEMMA 1.3.6. *Let $(\mu_k)_k$ be a sequence in $Y^p(D;\Xi)$, such that $\mu_k \rightharpoonup \mu$ p-weakly*. Then $\mathcal{T}_g(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ p-weakly*, for every $g \in L^p(X;\Xi)$.*

More in general

LEMMA 1.3.7. *Let $\mu_k \rightharpoonup \mu$ p-weakly* and $(g_k)_k$ be a bounded sequence in $L^p(D;\Xi)$ with $g_k \to g$ strongly in $L^1(D;\Xi)$. Then $\mathcal{T}_{g_k}(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ p-weakly*.*

PROOF. Since $(g_k)_k$ is bounded in $L^p(D;\Xi)$, the p-moments of $\mathcal{T}_{g_k}(\mu_k)$ are equibounded. We now prove that $\mathcal{T}_{g_k}(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ weakly*. Since $\mathcal{C}_0(D \times \Xi)$ is the closure of $\mathcal{C}_0^\infty(D\times\Xi)$ with respect to the norm $\|\cdot\|_\infty$, thanks to (1.3.2), to prove that $\mathcal{T}_{g_k}(\mu_k) \rightharpoonup \mathcal{T}_g(\mu)$ weakly* it is enough to show that

$$\int_{D\times\Xi} f(x, \xi + g_k(x))\ d\mu_k(x,\xi) \to \int_{D\times\Xi} f(x, \xi + g(x))\ d\mu(x,\xi),$$

for every $f \in \mathcal{C}_0^\infty(D \times \Xi)$.

Let $f \in \mathcal{C}_0^\infty(D \times \Xi)$, we have

$$\Big| \int_{D\times\Xi} f(x,\xi + g(x))\ d\mu(x,\xi) - \int_{D\times\Xi} f(x,\xi+g_k(x))\ d\mu_k(x,\xi) \Big| \le$$
$$\le \Big| \int_{D\times\Xi} f(x,\xi + g(x))\ d\mu(x,\xi) - \int_{D\times\Xi} f(x,\xi+g(x))\ d\mu_k(x,\xi) \Big| +$$
$$+ \int_{D\times\Xi} |f(x,\xi+g(x)) - f(x,\xi+g_k(x))|\ d\mu_k(x,\xi).$$

By the Lipschitz continuity of f and (1.3.2), the last line can be estimated by $c\|g - g_k\|_1$ for a positive constant c; Lemma 1.3.6 implies now the thesis. $\square$

If we deal with Young measures generated by gradients, the following Lemma, in the version of Fonseca, Müller, and Pedregal ([**20**, Lemma 1.2]), can be very useful.

LEMMA 1.3.8 (Decomposition Lemma). *Let $(v_j)_j$ be a bounded sequence in $H^1(D;\Xi)$. Then there exists a subsequence $(v_{j_k})_k$ of $(v_j)_j$, and another sequence $(w_k)_k$ bounded in $H^1(D;\Xi)$, such that*

$$\mathcal{L}^d(\{v_{j_k} \neq w_k \text{ or } \nabla v_{j_k} \neq \nabla w_k\}) \to 0, \tag{1.3.6}$$

as $k \to \infty$, and $(|\nabla w_k|^2)_k$ is equiintegrable.

Note that condition (1.3.6) implies that both sequences $(\nabla v_{j_k})_k$ and $(\nabla w_k)_k$ generate the same Young measure, i.e., $\boldsymbol{\delta}_{\nabla v_{j_k}}$ and $\boldsymbol{\delta}_{\nabla w_k}$converge to the same Young measure.

Using part of the arguments of [**20**] and a more careful diagonalization argument, it can be proved the following Lemma.

LEMMA 1.3.9. *Let $(v_j)_j$ be a bounded sequence in $L^2(D;\Xi)$ such that there exists a Young measure $\mu \in Y^2(D;\Xi)$ with $\delta_{v_j} \rightharpoonup \mu$ weakly*. Then there exists another sequence $(w_j)_j$, bounded in $L^2(D;\Xi)$, such that*

$$\mathcal{L}^d(\{v_j \neq w_j\}) \to 0, \tag{1.3.7}$$

as $j \to \infty$, and $(|w_j|^2)_j$ is equiintegrable.

The following theorem (see [**5**]) gives an important convergence result in case we deal with equiintegrable sequences.

THEOREM 1.3.10. (Fundamental Theorem for Young measures) *Given a Young measure μ, generated by a sequence of functions $(u_j)_j$, and a function $f \in \mathcal{C}(\Xi;\mathbb{R})$ such that the sequence $(f(u_j))_j$ is weakly sequentially relatively compact in $L^1(D)$, then*

$$f(u_j) \rightharpoonup f_\mu \quad \text{weakly in} \;\; L^1(D),$$

where the function $f_\mu \in L^1(D)$ is defined by $f_\mu(x) := \int_\Xi f(\xi)\, d\mu^x(\xi)$ for a.e. $x \in D$. In particular

$$\int_D f(u_j(x))\, dx \to \int_{D\times\Xi} f(\xi)\, d\mu(x,\xi).$$

The *barycentre* of a Young measure $\mu \in Y^p(D;\Xi)$ is the function $\mathrm{bar}(\mu) \in L^p(D;\Xi)$ defined as

$$\mathrm{bar}(\mu)(x) := \int_\Xi \xi\, d\mu^x(\xi),$$

for a.e. $x \in D$.

LEMMA 1.3.11. *Let μ_k be a sequence in $Y^2(D;\Xi)$, such that $\mu_k \rightharpoonup \mu$ 2-weakly*. Assume that there exists a sequence of functions $v_k \in H^1_0(D;\Xi)$ such that $\nabla v_k = \mathrm{bar}(\mu_k)$ for every k. Then there exists a unique function $v \in H^1_0(D;\Xi)$ such that $v_k \rightharpoonup v$ weakly in H^1 and $\nabla v = \mathrm{bar}(\mu)$.*

PROOF. Since $\|\nabla v_k\|_2^2 \le \int_{D\times\Xi} |\xi|^2\, d\mu_k(x,\xi)$ which is bounded uniformly with respect to k by hypothesis, using Poincaré inequality we can deduce that there exists a subsequence v_{k_h} and a function $v \in H^1_0(D;\Xi)$, such that $v_{k_h} \rightharpoonup v$ weakly in H^1. Using the definition of barycentre and Lemma 1.3.5, we deduce that $\nabla v = \mathrm{bar}(\mu)$ and hence $\nabla v_{k_h} \rightharpoonup \mathrm{bar}(\mu)$ weakly in $L^2(D;\Xi)$; together with Poincaré inequality, this implies that the whole sequence v_k converges to v weakly in H^1. $\square$

CHAPTER 2

Compatible systems of Young measures: the stochastic process formulation and the discrete case

In this chapter we present the definition of *compatible system of Young measures with finite p-moment* and the related notion of variation on a time interval. We propose an alternative formulation of Young measures and compatible systems using probabilistic language. Finally, we give a more explicit description of Young measures and compatible systems with values in a finite set.

The results of this chapter are contained in [**17**] and [**19**].

2.1. Compatible systems of Young measures

In this section we present the definitions and results related to the notion of compatible system of Young measures.

Let $2 \leq p \leq \infty$.

Definition 2.1.1. A *compatible system of Young measures with finite p-moments* on D with time set T and values in $\prod_{t\in T} \Xi_t$ is a family $\boldsymbol{\mu} = (\boldsymbol{\mu}_F)$ of Young measures $\boldsymbol{\mu}_F \in Y^p(D; \prod_{t\in F} \Xi_t)$, with F varying among all nonempty finite subsets of T, such that the compatibility condition

$$\tilde{\pi}^F_G(\boldsymbol{\mu}_F) = \boldsymbol{\mu}_G \tag{2.1.1}$$

is satisfied, for every nonempty finite subsets $G \subset F$ of T, where $\tilde{\pi}^F_G \colon D \times \prod_{t\in F} \Xi_t \to D \times \prod_{s\in G} \Xi_s$ maps $(x, (\xi_t)_{t\in F})$ in $(x, (\xi_s)_{s\in G})$.

The space of all such systems is denoted by $SY^p(D; \prod_{t\in T} \Xi_t)$; in the special case of $\Xi_t \equiv \Xi$, for every $t \in T$ we will use the notation $SY^p(T, D; \Xi)$.

Remark 2.1.2. Let $Y^p(D; \Xi)^T$ denote the set of all families of Young measures with finite p-moments on D with values in Ξ, indexed on the set T. Given $(\mu_t)_{t\in T} \in Y^p(D; \Xi)^T$, we can always construct a compatible system $\boldsymbol{\mu} \in SY^p(T, D; \Xi)$ satisfying $\boldsymbol{\mu}_t = \mu_t$ for every $t \in T$. Indeed it is enough to define

$$\boldsymbol{\mu}_F := \int_D \big(\bigotimes_{t\in F} \mu^x_t \big) \, dx,$$

for every nonempty finite subset F of T.

The space $SY^p(D; \prod_{t\in T} \Xi_t)$ will be equipped with the weakest topology for which the maps $\boldsymbol{\mu} \mapsto \boldsymbol{\mu}_F$ from $SY^p(D; \prod_{t\in T} \Xi_t)$ into $Y^p(D; \prod_{t\in F} \Xi_t)$, endowed with the weak* topology of $M_b(D \times \prod_{t\in F} \Xi_t)$, are continuous for every nonempty finite subset F of T. We will refer to this topology as the weak* topology of $SY^p(D; \prod_{t\in T} \Xi_t)$.

From now on we consider the case in which the time set is an interval $[0,T]$.

The following two definitions concern different approximation properties with respect to time.

DEFINITION 2.1.3. A compatible system of Young measures with finite p-moments $\boldsymbol{\mu}$ is said to be *left continuous* if for every finite sequence $t_1,\dots,t_m$ in $[0,T]$, with $t_1<\dots<t_m$, the following property holds:

$$\boldsymbol{\mu}_{s_1\dots s_m} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_m} \quad p\text{-weakly*}$$

as $s_i\to t_i$, with $s_i\in[0,T]$ and $s_i\le t_i$. We will denote the space of all left continuous compatible systems by $SY^p_-([0,T],D;\Xi)$.

DEFINITION 2.1.4. Given a subset Θ of $[0,T]$ with $\mathcal{L}^1([0,T]\setminus\Theta)=0$, a family of Young measures $\boldsymbol{\nu}\in Y^p(D;\mathbb{R}^M)^{[0,T]}$ is said to be Θ-*p-weakly* approximable from the left* if for every $t\in[0,T]\setminus\Theta$ there exists a sequence s^j in Θ converging to t, with $s^j\le t$, such that

$$\boldsymbol{\nu}_{s^j} \rightharpoonup \boldsymbol{\nu}_t \quad p\text{-weakly*} \tag{2.1.2}$$

as $j\to\infty$.

REMARK 2.1.5. Note that the notion of p-weakly* left continuity is much stronger than Θ-p-weakly* approximability from the left: indeed the first one requires that the convergence condition is satisfied not only for a single time but for every finite sequence of times, and does not depend on the choice of the sequence s_i^j approximating t_i.

The following theorem can be considered as a version of Helly's Theorem.

THEOREM 2.1.6. *Let $\boldsymbol{\mu}^k$ be a sequence in $SY^p([0,T],D;\Xi)$ such that*

$$\sup_k \operatorname{Var}(\boldsymbol{\mu}^k;0,T)\le C\,, \tag{2.1.3}$$

$$\sup_{t\in[0,T]}\sup_k \int_{D\times\Xi}|\xi|^p\,d\boldsymbol{\mu}^k_t(x,\xi)\le C^*\,, \tag{2.1.4}$$

for finite constants C and C^*. Then there exist a subsequence, still denoted by $\boldsymbol{\mu}^k$, a set $\Theta\subset[0,T]$, containing 0 and such that $[0,T]\setminus\Theta$ is at most countable, and $\boldsymbol{\mu}\in SY^p_-([0,T],D;\Xi)$ with

$$\operatorname{Var}(\boldsymbol{\mu};0,\mathrm{T})\le \mathrm{C}\,, \tag{2.1.5}$$

$$\textstyle\int_{D\times\Xi}|\xi|^p\,d\boldsymbol{\mu}_t(x,\xi)\le C^* \quad \textit{for every } t\in[0,T]\,, \tag{2.1.6}$$

such that, for every nonempty finite subset F of Θ, we have

$$\boldsymbol{\mu}^k_F \rightharpoonup \boldsymbol{\mu}_F \quad p\text{-weakly*}. \tag{2.1.7}$$

The proof of this Theorem follows easily from Theorem 8.10 of [**10**], since every Young measure can be seen as a generalized Young measure and our more restrictive hypotheses force the limit to be an element of $SY^p([0,T],D;\Xi)$.

We define the *variation* of $\boldsymbol{\mu}\in SY^2([0,T],D;\Xi)$ on $[a,b]\subseteq[0,T]$ by

$$\operatorname{Var}(\boldsymbol{\mu};a,b):=\sup\sum_{i=1}^k\int_{D\times\Xi^{k+1}}|\xi_i-\xi_{i-1}|\,d\boldsymbol{\mu}_{t_0\dots t_k}(x,\xi_0,\dots,\xi_k)$$

where the supremum is taken over all finite partitions $a = t_0 < \cdots < t_k = b$ of the interval $[a, b]$ (with the convention $\mathrm{Var}(\boldsymbol{\mu}; a, b) = 0$, if $a = b$).

If $H : \Xi \to [0, +\infty)$ is positively homogeneous of degree one and satisfies the triangle inequality, we can define the H-variation of $\boldsymbol{\mu} \in SY^2([0, T], D; \Xi)$ on the time interval $[a, b] \subseteq [0, T]$ as

$$\mathrm{Var}_H(\boldsymbol{\mu}; a, b) := \sup \sum_{i=1}^{k} \int_{D \times \Xi^{k+1}} H(\xi_i - \xi_{i-1})\, d\boldsymbol{\mu}_{t_0 \ldots t_k}(x, \xi_0, \ldots, \xi_k), \tag{2.1.8}$$

where the supremum is taken over all finite partitions $a = t_0 < \cdots < t_k = b$ of the interval $[a, b]$ (with the convention $\mathrm{Var}_H(\boldsymbol{\mu}; a, b) = 0$, if $a = b$).

Adapting the argument in [**10**] (Theorem 8.11), it can be proved the following Lemma.

Lemma 2.1.7. *Let $\boldsymbol{\mu}^k$ be a sequence in $SY^2([0, T], D; \Xi)$. Suppose that there exist a dense set $\Theta \subset [0, T]$ containing 0 and $\boldsymbol{\mu} \in SY^2_{-}([0, T], D; \Xi)$, such that*

$$\boldsymbol{\mu}^k_F \rightharpoonup \boldsymbol{\mu}_F \quad \text{2-weakly}^*$$

for every nonempty finite subset F in Θ; then

$$\mathrm{Var}_H(\boldsymbol{\mu}; 0, T) \leq \liminf_{k \to \infty} \mathrm{Var}_H(\boldsymbol{\mu}^k; 0, T)$$

for every positively one homogeneous function $H : \Xi \to [0, +\infty)$ satisfying the triangle inequality.

Definition 2.1.8. Fix a finite sequence $0 = t_1 < \cdots < t_m = T$ in $[0, T]$. For every $\mu \in Y^2(D; \Xi^m)$, it is possible to define the *piecewise constant interpolation* $\boldsymbol{\mu}^{pwc} \in SY^2([0, T], D; \Xi)$ in the following way. For every finite sequence $\tau_1 < \cdots < \tau_n$ of elements of $[0, T]$ let $\rho_{\tau_1 \ldots \tau_n} : D \times \Xi^m \times \mathbb{R} \to D \times \Xi^n \times \mathbb{R}$ be defined by

$$\rho_{\tau_1 \ldots \tau_n}(x, \xi_{t_1}, \ldots, \xi_{t_m}) := (x, \xi_{\tau_1}, \ldots, \xi_{\tau_n}),$$

with $\xi_{\tau_i} = \xi_{t_j}$, where j is the largest index such that $t_j \leq \tau_i$. The compatible system of Young measures with finite second moments $\boldsymbol{\mu}^{pwc}$ is then defined by

$$\boldsymbol{\mu}^{pwc}_{\tau_1 \ldots \tau_n} := \rho_{\tau_1 \ldots \tau_n}(\mu).$$

Lemma 2.1.9. *Let $(\boldsymbol{\mu}^n)_n$ be a sequence in $SY^2([0, T], D; \Xi_1)$ and $(\boldsymbol{\nu}^n)_n$ a sequence in $SY^2([0, T], D; \Xi_1 \times \Xi_2)$. Assume that they satisfy $\tilde{\pi}_{\Xi_1}(\boldsymbol{\nu}^n_t) = \boldsymbol{\mu}^n_t$, for every $t \in [0, T]$, that*

$$\sup_{t \in [0,T]} \sup_n \int_{D \times \Xi_1 \times \Xi_2} |(\xi_1, \xi_2)|^2 \, d\boldsymbol{\nu}^n_t(x, \xi_1, \xi_2) \leq C, \tag{2.1.9}$$

for a positive constant C, and that there exist a subset Θ of $[0, T]$, containing 0, with $\mathcal{L}^1([0, T] \setminus \Theta) = 0$, and $\boldsymbol{\mu} \in SY^2_{-}([0, T], D; \Xi_1)$ with

$$\boldsymbol{\mu}^n_{t_1 \ldots t_m} \rightharpoonup \boldsymbol{\mu}_{t_1 \ldots t_m}, \quad \text{2-weakly}^*,$$

for every $t_1 < \cdots < t_m$ in Θ. For every $t \in \Theta$ let $(n^t_k)_k$ be an increasing sequence of integers; then there exists $\boldsymbol{\nu} \in SY^2([0, T], D; \Xi_1 \times \Xi_2)$, such that $\tilde{\pi}_{\Xi_1}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$, for every $t \in [0, T]$ and satisfying the following properties:

(1) *for every $t \in \Theta$, there exists a subsequence $\boldsymbol{\nu}_t^{n^t_{k,i}}$ of $\boldsymbol{\nu}_t^{n^t_k}$ such that*

$$\boldsymbol{\nu}_t^{n^t_{k,i}} \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly}^*; \tag{2.1.10}$$

(2) *for every* $t \in [0,T] \setminus \Theta$, *there exists a sequence* s^j *in* Θ, *converging to* t, *with* $s^j \leq t$, *such that*

$$\boldsymbol{\nu}_{s^j} \rightharpoonup \boldsymbol{\nu}_t \quad \textit{2-weakly*}. \tag{2.1.11}$$

The proof of this Lemma was contained in a preliminary version of [**12**].

Proof. Fix $t \in \Theta$; thanks to (2.1.9), we can deduce that there exist $\nu_t^0 \in Y^2(D; \Xi_1 \times \Xi_2)$ and a subsequence $(\boldsymbol{\nu}_t^{n_{k,i}^t})_i$ of $(\boldsymbol{\nu}_t^{n_k^t})_k$, satisfying (2.1.10). Thanks to Remark 1.3.4 and Lemma 1.3.5, for every $t \in \Theta$ we have

$$\int_{D \times \Xi_1 \times \Xi_2} |(\xi_1, \xi_2)|^2 \, d\nu_t^0(x, \xi_1, \xi_2) \leq C \tag{2.1.12}$$

and $\tilde{\pi}_{\Xi_1}(\nu_t^0) = \boldsymbol{\mu}_t$.

Consider now the sets B_t defined in the following way:

- if $t \in \Theta$, B_t denotes the collection of all $\boldsymbol{\nu} \in SY^2([0,T], D; \Xi_1 \times \Xi_2)$ such that the second moments of $\boldsymbol{\nu}_s$ are bounded by the constant C appearing in (2.1.9), for every $s \in [0,T]$, and satisfying $\boldsymbol{\nu}_t = \nu_t^0$;
- if $t \notin \Theta$, B_t is the collection of all $\boldsymbol{\nu} \in SY^2([0,T], D; \Xi_1 \times \Xi_2)$, such that the second moments of $\boldsymbol{\nu}_s$ are bounded by the constant C appearing in (2.1.9), for every $s \in [0,T]$, and for which there exists a sequence s^j in Θ, converging to t with $s^j \leq t$, such that $\nu_{s^j}^0 \rightharpoonup \boldsymbol{\nu}_t$, weakly*.

For every $t \in [0,T]$, $B_t \neq \emptyset$: indeed, if $t \in \Theta$ it comes immediately from Remark 2.1.2 applied to $(\boldsymbol{\nu}_s)_{s \in [0,T]} \in Y^2(D; \Xi)^{[0,T]}$ defined by $\boldsymbol{\nu}_s := \nu_t^0$ for every $s \in [0,T]$; if $t \notin \Theta$, thanks to (2.1.12), there exist s^j, with $s^j \in \Theta$, $s^j \to t$ and $s^j \leq t$, and $\mu \in Y^2(D; \Xi_1 \times \Xi_2)$, such that $\nu_{s^j}^0 \rightharpoonup \mu$ 2-weakly*, hence the second moment of μ is bounded by C and we can apply Remark 2.1.2 to $(\boldsymbol{\nu}_s)_{s \in [0,T]} \in Y^2(D; \Xi)^{[0,T]}$ defined by $\boldsymbol{\nu}_s := \mu$, for every $s \in [0,T]$, and find an element of B_t. Using Remark 1.3.2 we can see that the set of all Young measures μ for which there exists a sequence $s^j \to t$ with $s^j \leq t$ and $\nu_{s^j}^0 \rightharpoonup \mu$ weakly* is sequentially closed with respect to the weak* topology (thanks to (2.1.12)); moreover we observe that the set of all $\boldsymbol{\nu} \in SY^2([0,T], D; \Xi)$ with the second moments equibounded by the constant C is closed in the weak* topology of $SY^2([0,T], D; \Xi)$, therefore, using again Remark 1.3.2, we can conclude that, for every $t \in [0,T]$, B_t are closed subsets of $SY^2([0,T], D; \Xi_1 \times \Xi_2)$, endowed with the weak* topology. Moreover the family has the finite intersection property (for every finite sequence $t_1 < \cdots < t_m$ in $[0,T]$, using Definition 2.1.8, we can find an element belonging to $B_{t_1} \cap \cdots \cap B_{t_m}$) and is contained in the set of all $\boldsymbol{\nu} \in SY^2([0,T], D; \Xi_1 \times \Xi_2)$ for which the second moments of $\boldsymbol{\nu}_t$ are uniformly bounded by the constant C appearing in (2.1.9); since, thanks to Tychonoff's Theorem, this is a compact subset of $SY^2([0,T], D; \Xi_1 \times \Xi_2)$, endowed with the weak* topology, we can conclude that there exists $\boldsymbol{\nu}$ belonging to B_t, for every $t \in [0,T]$. By construction $\boldsymbol{\nu}$ satisfies (1) and (2) and from the left continuity of $\boldsymbol{\mu}$ we can deduce that $\tilde{\pi}_{\Xi_1}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$, for every $t \in [0,T]$, as required. □

2.2. The probabilistic formulation

In the first part of this section we want to point out that Young measures can be presented using a probabilistic language, and precisely the notion of random variable.

While for a single Young measure this probabilistic presentation does not introduce relevant simplifications, it will be very useful in the case of families of time-dependent Young measures.

For simplicity, in this section we assume $|D| = 1$.

Probability spaces of the form $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$, where Ω is a measurable space, $\mathcal{B}(D)$ denotes the Borel σ-algebra on D, $\mathcal{F}$ is a σ-algebra on Ω and P a probability measure with the property $\pi_D(P) = \mathcal{L}^d$, will be called *$(D, \mathcal{L}^d)$-probability spaces.*

We can associate to every Young measure μ on D with values in Ξ a random variable Y defined on a $(D, \mathcal{L}^d)$-probability space with values in Ξ, in such a way that

$$\int_{D\times\Xi} f(x,\xi)\; d\mu(x,\xi) = \int_{D\times\Omega} f(x, Y(x,\omega))\; dP(x,\omega), \tag{2.2.1}$$

for every bounded Borel function $f\colon D \times \Xi \to \mathbb{R}$. Indeed it is enough to take as Ω the space Ξ itself, as Y the projection on Ξ, and as P the measure μ itself, which is a probability measure thanks to the assumption on $|D|$ and (1.3.2). Conversely, given any Ξ-valued random variable Y on a $(D, \mathcal{L}^d)$-probability space, formula (2.2.1) defines a Young measure μ, which will be denoted by $(\pi_D, Y)(P)$, since it coincides with the image of the measure P under the map $(\pi_D, Y)\colon D \times \Omega \to D \times \Xi$.

We say that a random variable Y on a $(D, \mathcal{L}^d)$-probability space $(D \times \Omega, P)$ has finite p-moment if

$$\int_{D\times\Omega} |Y(x,\omega)|^p\; dP(x,\omega) < \infty.$$

Hence a Young measure has finite p-moment if and only if the associated random variable does.

In the particular case of $\mu = \boldsymbol{\delta}_u \in Y^p(D;\Xi)$ with $u \in L^p(D;\Xi)$, for every $(D, \mathcal{L}^d)$-probability space we can associate to $\boldsymbol{\delta}_u$ the random variable $Y\colon D \times \Omega \to \Xi$, defined by $Y(x,\omega) := u(x)$ for $\mathcal{L}^d$-a.e. $x \in D$ and for every $\omega \in \Omega$; we will denote this random variable simply by u.

In the more general case of a measure $\mu \in Y^p(D;\Xi)$ which is not associated to a function, there always exists a bounded sequence of functions $u_k \in L^p(D;\Xi)$ such that

$$\boldsymbol{\delta}_{u_k} \rightharpoonup \mu \quad p\text{-weakly*},$$

as $k \to \infty$ (see [**43**, Theorem 7.7 pg. 126]). In probabilistic language this means that the random variable $Y \in L^p(D \times \Omega;\Xi)$ satisfies the following property:

$$(\pi_D, u_k)(P) \rightharpoonup (\pi_D, Y)(P) \quad p\text{-weakly*},$$

as $k \to \infty$, or equivalently

$$\frac{\mathcal{L}^d(\{x \in B \quad u_k(x) \in A\})}{\mathcal{L}^d(B)} \to \frac{P(\{(x,\omega) \in B \times \Omega \;:\; Y(x,\omega) \in A\})}{P(B \times \Omega)}, \tag{2.2.2}$$

as $k \to \infty$, for every Borel sets $B \subset\subset D$ and $A \subset\subset \mathbb{R}$ with $P(\partial(\{(x,\omega) \;:\; x \in B,\ Y(x,\omega) \in A\})) = 0$. Hence the dependence of Y on the variable ω plays the role of describing the statistics of the oscillations of the sequence u_k.

As we have seen, if we deal with a single random variable the association to a Young measure is immediate; more complicated is the case of a stochastic process $(\boldsymbol{Y}_t)_{t\in T}$ on a $(D, \mathcal{L}^d)$-probability space in a time set T: indeed the family of measures $((\pi_D, \boldsymbol{Y}_t)(P))_{t\in T}$

gives an insufficient information on the stochastic process, since in general we cannot go back to $(\pi_D, \boldsymbol{Y}_{t_1}, \dots, \boldsymbol{Y}_{t_n})(P)$, for an increasing sequence of time instants $t_1 < \cdots < t_n$, just using $(\pi_D, \boldsymbol{Y}_{t_i})(P)$, $i = 1, \dots, n$.

In the second part of this section, using a modification of Kolmogorov Theorem (see [**26**, pg. 29]), we want to show that the correct correspondence is between stochastic processes and compatible systems of Young measures.

Given a stochastic process $(\boldsymbol{Y}_t)_{t\in T}$ on a $(D, \mathcal{L}^d)$-probability space, with

$$\boldsymbol{Y}_t \in L^p(D \times \Omega; \Xi_t),$$

we can define a family of Young measures on D, indexed by the nonempty finite subsets F of T, as

$$\boldsymbol{\mu}_F := (\pi_D, (\boldsymbol{Y}_t)_{t\in F})(P). \tag{2.2.3}$$

It is immediate to see that every $\boldsymbol{\mu}_F$ has finite p-moment and that this family satisfies the compatibility condition (2.1.1).

The following remark is technical and will be used to prove the correspondence between compatible systems and stochastic processes.

Remark 2.2.1. If $\boldsymbol{\mu}$ satisfies the compatibility condition, for every nonempty finite subsets $G \subset F$ of T there exists a set N_G^F of D with $\mathcal{L}^d(N_G^F) = 0$, such that

$$\pi_G^F(\boldsymbol{\mu}_F^x) = \boldsymbol{\mu}_G^x \quad \text{for every } x \in D \setminus N_G^F. \tag{2.2.4}$$

Conversely, if (2.2.4) holds for a.e. $x \in D$, then $\boldsymbol{\mu}$ satisfies the compatibility condition (2.1.1).

Hence, up to subsets of D with zero measure, the compatibility condition commutes in some sense with the disintegration.

In the next Theorem we will show that to every compatible system of Young measures with finite p-moments we can associate a stochastic process on a suitable $(D, \mathcal{L}^d)$-probability space.

Theorem 2.2.2. *Given a set of indices T and a compatible system $\boldsymbol{\mu} \in SY^p(D; \prod_{t\in T} \Xi_t)$, there exist a $(D, \mathcal{L}^d)$-probability space and a stochastic process $(\boldsymbol{X}_t)_{t\in T}$ with*

$$\boldsymbol{X}_t \in L^p(D \times \Omega; \Xi_t), \tag{2.2.5}$$

for every $t \in T$, such that

$$(\pi_D, (\boldsymbol{X}_t)_{t\in F})(P) = \boldsymbol{\mu}_F, \tag{2.2.6}$$

for every nonempty finite subset F of T.

Proof. Let t_∞ be an index such that $t_\infty \notin T$. Set $\hat{T} := \{t_\infty\} \cup T$ and let $\Xi_{t_\infty} := D$.

We want to construct a system of finite dimensional distributions indexed on $\hat{T}$, satisfying the hypotheses of Kolmogorov Theorem.

Let F be a nonempty finite subset of $\hat{T}$; we distinguish three cases. If $t_\infty \in F$ and $F \setminus \{t_\infty\} \neq \emptyset$ we set

$$\boldsymbol{\nu}_F = \boldsymbol{\mu}_{F\setminus\{t_\infty\}}. \tag{2.2.7}$$

If $t_\infty \notin F$, we set

$$\boldsymbol{\nu}_F := \pi_{\prod\limits_{\tau\in F} \Xi_\tau}(\boldsymbol{\mu}_F), \tag{2.2.8}$$

where $\pi_{\prod_{\tau\in F}\Xi_\tau}: D\times\prod_{\tau\in F}\Xi_\tau \to \prod_{\tau\in F}\Xi_\tau$ denotes the usual projection. Finally, if $F = \{t_\infty\}$, we set

$$\boldsymbol{\nu}_F := \mathcal{L}^d. \tag{2.2.9}$$

The system of finite dimensional distributions $\boldsymbol{\nu}$ satisfies the following compatibility condition: for every nonempty finite subsets $G \subset F$ of $\hat{T}$, we have

$$\pi^F_G(\boldsymbol{\nu}_F) = \boldsymbol{\nu}_G. \tag{2.2.10}$$

Indeed, in the case $t_\infty \in F$ and $F\setminus\{t_\infty\} \neq \emptyset$, it follows from compatibility condition for $\boldsymbol{\mu}$ if $t_\infty \in G$ and $G\setminus\{t_\infty\}\neq\emptyset$, it comes from (1.3.2) if $G = \{t_\infty\}$, and it easily follows from (2.2.8) if $t_\infty \notin G$.In case $t_\infty \notin F$, (2.2.10) can be proved using the construction in (2.2.8) and the analysis of the previous case.

By (2.2.10), $\boldsymbol{\nu}$ satisfies the hypotheses of Kolmogorov Theorem; therefore it is enough to choose $\Omega := \prod_{t\in T}\Xi_t$, $\mathcal{F}$ the product of the Borel σ-algebras of Ξ_t, for $t\in T$, $\boldsymbol{X}_t: \Omega\to\Xi_t$, for $t \in T$, the usual projections, and Kolmogorov Theorem guarantees the existence of a probability measure P on $(D\times\Omega, \mathcal{B}(D)\times\mathcal{F})$ with $\pi_D(P) = \mathcal{L}^d$ which satisfies (2.2.6). $\square$

Using the previous result we will prove that we can associate to a pair of two compatible systems, connected by a further compatibility condition, a pair of stochastic processes on the same probability space.

LEMMA 2.2.3. *Let V and W finite dimensional Hilbert spaces, $\mu \in Y^p(D;V)$ and $\nu \in Y^p(D; V\times W)$ be such that*

$$\tilde{\pi}_V(\nu) = \mu. \tag{2.2.11}$$

Then for $\mathcal{L}^d$-a.e. $x \in D$ we have

$$(\nu^x)^v = \nu^{(x,v)},$$

for μ^x-a.e. $v\in V$, where $(\nu^x)^v$ is the disintegration of ν^x with respect to μ^x and $\nu^{(x,v)}$ the disintegration of ν with respect to μ.

PROOF. It is easy to see, as for Remark 2.2.1, that there exists a set $N\subseteq D$, with $\mathcal{L}^d(N) = 0$, such that for every $x \in D\setminus N$

$$\pi_V(\nu^x) = \mu^x. \tag{2.2.12}$$

Hence for every bounded Borel function $f: D\times V\times W\to\mathbb{R}$,

$$\int_{D\times V\times W} f(x,v,w)\ d\nu(x,v,w) = \int_D\Big(\int_{V\times W} f(x,v,w)\ d\nu^x(v,w)\Big)\ dx =$$
$$= \int_D\Big(\int_V\Big(\int_W f(x,v,w)\ d(\nu^x)^v(w)\Big)\ d\mu^x(v)\Big)\ dx.$$

On the other hand, thanks to (2.2.11),

$$\int_{D\times V\times W} f(x,v,w)\ d\nu(x,v,w) = \int_{D\times V}\Big(\int_W f(x,v,w)\ d\nu^{(x,v)}(w)\Big)\ d\mu(x,v) =$$
$$= \int_D\Big(\int_V\Big(\int_W f(x,v,w)\ d\nu^{(x,v)}(w)\Big)\ d\mu^x(v)\Big)\ dx.$$

This concludes the proof. $\square$

THEOREM 2.2.4. *Let T a set of indices, V_t and W_t finite dimensional Hilbert spaces, for every $t \in T$. Let $\boldsymbol{\mu} \in SY^p(D;(V_t)_{t\in T})$ and $\boldsymbol{\nu} \in SY^p(D;(V_t \times W_t)_{t\in T})$. Assume that*

$$\tilde{\pi}_{V_t}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t,$$

for every $t \in T$. Then there exist a $(D,\mathcal{L}^d)$-probability space $(D\times\Omega, \mathcal{B}(D)\otimes\mathcal{F}, P)$ and a stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in T}$ with

$$\boldsymbol{Z}_t \in L^p(D\times\Omega; V_t),$$
$$\boldsymbol{Y}_t \in L^p(D\times\Omega; W_t),$$

for every $t \in T$, such that

$$(\pi_D, (\boldsymbol{Z}_t)_{t\in F})(P) = \boldsymbol{\mu}_F, \tag{2.2.13}$$

for every nonempty finite subset F of T, and

$$(\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) = \boldsymbol{\nu}_t, \tag{2.2.14}$$

for every $t \in T$.

PROOF. We want to construct from $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ a unique compatible system and to apply to it Theorem 2.2.2.

Fix a nonempty finite subset F of T. Denote by $(\boldsymbol{\mu}_F^x)_{x\in D}$, $(\boldsymbol{\nu}_t^x)_{x\in D}$ the disintegrations with respect to $\mathcal{L}^d$ of $\boldsymbol{\mu}_F$, and $\boldsymbol{\nu}_t$, $t \in F$, respectively. As observed in (2.2.12), $\pi_{V_t}(\boldsymbol{\nu}_t^x) = \boldsymbol{\mu}_t^x$, for a.e. $x \in D$, for every $t \in F$. Hence, for a.e. $x \in D$, we can write

$$\boldsymbol{\nu}_t^x = \int_{V_t} (\boldsymbol{\nu}_t^x)^{v_t}\, d\boldsymbol{\mu}_t^x(v_t).$$

Using the fact that the disintegration is a measurable family, a Dynkin class argument, and Lemma 2.2.3, we can deduce that

$$(x,(v_t)_{t\in F}) \mapsto \big(\bigotimes_{t\in F} (\boldsymbol{\nu}_t^x)^{v_t}\big)(B) \tag{2.2.15}$$

is a Borel measurable function, for every Borel subset B of $\prod_{t\in F} W_t$. In particular for a.e. $x \in D$ the function $(v_t)_{t\in F} \mapsto (\otimes_{t\in F}(\boldsymbol{\nu}_t^x)^{v_t})(B)$ is Borel measurable; hence, for every Borel sets $A \subseteq \prod_{t\in F} V_t$ and $B \subseteq \prod_{t\in F} W_t$, we can define a measure $\tilde{\boldsymbol{\nu}}_F^x$ on $\prod_{t\in F}(V_t \times W_t)$ by

$$\tilde{\boldsymbol{\nu}}_F^x(A\times B) := \int_A \big(\bigotimes_{t\in F} (\boldsymbol{\nu}_t^x)^{v_t}\big)(B)\, d\boldsymbol{\mu}_F^x((v_t)_{t\in F}), \tag{2.2.16}$$

for a.e. $x \in D$.

By construction, for a.e. $x \in D$, $\tilde{\boldsymbol{\nu}}_F^x$ is a probability measure with the properties

$$\pi_{\prod_{t\in F} V_t}(\tilde{\boldsymbol{\nu}}_F^x) = \boldsymbol{\mu}_F^x, \tag{2.2.17}$$

$$\pi_{V_t\times W_t}(\tilde{\boldsymbol{\nu}}_F^x) = \boldsymbol{\nu}_t^x, \tag{2.2.18}$$

for every $t \in F$.

The Borel measurability of the function in (2.2.15) guarantees that $(\tilde{\boldsymbol{\nu}}_F^x)_{x\in D}$ is a measurable family of probability measures on $\prod_{t\in F}(V_t \times W_t)$ and we can define

$$\tilde{\boldsymbol{\nu}}_F := \int_D \tilde{\boldsymbol{\nu}}_F^x\, dx.$$

It is easy to check that, for F running over all nonempty finite subsets of T, $\tilde{\boldsymbol{\nu}}_F \in Y^p(D;\prod_{t\in F}(V_t \times W_t))$ and satisfies the compatibility condition; hence we have $\tilde{\boldsymbol{\nu}} \in SY^p(D;(V_t \times W_t)_{t\in T})$. Moreover, thanks to (2.2.17) and (2.2.18),

$$\tilde{\pi}_{\prod_{t\in F} V_t}(\tilde{\boldsymbol{\nu}}_F) = \boldsymbol{\mu}_F, \tag{2.2.19}$$

for every nonempty finite subset F of T and

$$\tilde{\pi}_{V_t\times W_t}(\tilde{\boldsymbol{\nu}}_t) = \boldsymbol{\nu}_t, \tag{2.2.20}$$

for every $t \in T$.

Applying Theorem 2.2.2 to $\tilde{\boldsymbol{\nu}}$, we obtain a $(D,\mathcal{L}^d)$-probability space $(D\times\Omega,\mathcal{B}(D)\otimes\mathcal{F},P)$ and a stochastic process $(\boldsymbol{Z}_t,\boldsymbol{Y}_t)_{t\in T}$, with $(\boldsymbol{Z}_t,\boldsymbol{Y}_t)\colon D\times\Omega \to V_t\times W_t$, such that

$$(\pi_D,(\boldsymbol{Y}_t,\boldsymbol{Z}_t)_{t\in F})(P) = \tilde{\boldsymbol{\nu}}_F, \tag{2.2.21}$$

for every nonempty finite subset F of T. Using the construction of $(\boldsymbol{Z}_t,\boldsymbol{Y}_t)$ and (2.2.19), (2.2.20), we can obtain the thesis. $\square$

If the time set is an interval $[0,T]\subset\mathbb{R}$ and $\Xi_t \equiv \Xi$ for every $t\in[0,T]$, the notion of variation of a stochastic process $(\boldsymbol{Y}_t)_{t\in[0,T]}$ on a $(D,\mathcal{L}^d)$-probability space, with $\boldsymbol{Y}_t \in L^1(D\times\Omega;\Xi)$, is defined in the usual way: for every time interval $[a,b]\subseteq[0,T]$ we set

$$\begin{aligned}\mathrm{Var}(\boldsymbol{Y},P;a,b) :&= \sup\sum_{i=1}^{k}\int_{D\times\Omega}|\boldsymbol{Y}_{t_i}(x,\omega)-\boldsymbol{Y}_{t_{i-1}}(x,\omega)|\,dP(x,\omega) = \\ &= \sup\sum_{i=1}^{k}\|\boldsymbol{Y}_{t_i}-\boldsymbol{Y}_{t_{i-1}}\|_1,\end{aligned}$$

where the supremum is taken over all finite partitions $a = t^0 < \cdots < t^k = b$ of the interval $[a,b]$ (with the convention $\mathrm{Var}(\boldsymbol{Y},P;a,b)=0$, if $a=b$).

Therefore, if $\boldsymbol{\mu}$ is the compatible system associated to the stochastic process $\boldsymbol{Y}$ we have $\mathrm{Var}(\boldsymbol{Y},P;a,b) = \mathrm{Var}(\boldsymbol{\mu};a,b)$.

Before closing this section, we rephrase the notions introduced in Definition 2.1.3 and Definition 2.1.4 in terms of stochastic processes.

Definition 2.2.5. A stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ defined on a $(D,\mathcal{L}^d)$-probability space $(D\times\Omega,\mathcal{B}(D)\otimes\mathcal{F},P)$ is said to be *p-weakly* left continuous* if for every finite sequence $t_1<\cdots<t_n$ in $[0,T]$ we have

$$(\pi_D,\boldsymbol{X}_{s_1^j},\ldots,\boldsymbol{X}_{s_n^j})(P) \rightharpoonup (\pi_D,\boldsymbol{X}_{t_1},\ldots,\boldsymbol{X}_{t_n})(P) \quad p\text{-weakly*}$$

as $j\to\infty$, whenever $s_i^j \to t_i$ and $s_i^j \le t_i$ for $i=1,\ldots,n$.

Definition 2.2.6. Given a subset Θ of $[0,T]$ satisfying $\mathcal{L}^1([0,T]\setminus\Theta)=0$, a stochastic process $(\boldsymbol{X}_t)_{t\in[0,T]}$ defined on a $(D,\mathcal{L}^d)$-probability space $(D\times\Omega,\mathcal{B}(D)\otimes\mathcal{F},P)$ is said to be Θ-*p-weakly* approximable from the left* if for every $t\in[0,T]\setminus\Theta$ there exists a sequence s^j in Θ converging to t, with $s^j\le t$ and

$$(\pi_D,\boldsymbol{X}_{s^j})(P) \rightharpoonup (\pi_D,\boldsymbol{X}_t)(P) \quad p\text{-weakly*} \tag{2.2.22}$$

as $j\to\infty$.

2.3. The discrete case

Let Z be a finite subset of $\mathbb{R}^m$.

The space of Young measures on D with values in Z is indicated with $Y(D;Z)$ and the space of compatible systems on D with time set A and values in Z is denoted by $SY(A,D;Z)$.

It is easy to see that $\mu \in Y(D;Z)$ if and only if its disintegration $(\mu^x)_{x\in D}$ can be written as

$$\mu = \sum_{\alpha=1}^{q} b_\alpha \boldsymbol{\delta}_{\theta_\alpha}, \tag{2.3.1}$$

where b_α are functions in $L^\infty(D;[0,1])$ satisfying the condition

$$\sum_{\alpha=1}^{q} b_\alpha(x) = 1, \quad \text{for a.e. } x \in D. \tag{2.3.2}$$

In disintegrated form, formula (2.3.1) can be written as

$$\boldsymbol{\mu}^x = \sum_{\alpha=1}^{q} b_\alpha(x)\delta_{\theta_\alpha} \quad \text{for a.e. } x \in D.$$

Therefore $Y(D;Z)$ can be identified with the set of all families $b = (b_\alpha)_{\alpha=1}^q$ in $L^\infty(D;[0,1])$ satisfying condition (2.3.2).

Set $\mathscr{A}_q^n := \{1,\dots,q\}^n$. If $\boldsymbol{\mu} \in SY(A,D;Z)$, then for every $t_1 < \cdots < t_n$ in A there exists a finite family $\boldsymbol{b}^{t_1\dots t_n} = (\boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n})_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_n^q}$ in $L^\infty(D;[0,1])$, satisfying the property

$$\sum_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n} \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n}(x) = 1, \quad \text{for a.e. } x \in D, \tag{2.3.3}$$

and such that

$$\boldsymbol{\mu}_{t_1\dots t_n} = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n} \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n}\boldsymbol{\delta}_{(\theta_{\alpha_1},\dots,\theta_{\alpha_n})}, \tag{2.3.4}$$

for every finite sequence $t_1 < \cdots < t_n$ in A.

The projection property of compatible systems can be formulated in a simpler way using this language: given any finite sequence $t_1 < \cdots < t_n$ in A ,we have

$$\boldsymbol{b}^{t_1\dots t_{i-1}t_{i+1}\dots t_n}_{\alpha_1\dots\alpha_{i-1}\alpha_{i+1}\dots\alpha_n} = \sum_{\beta=1}^{q} \boldsymbol{b}^{t_1\dots t_{i-1}t_it_{i+1}\dots t_n}_{\alpha_1\dots\alpha_{i-1}\beta\alpha_{i+1}\dots\alpha_n}, \tag{2.3.5}$$

a.e. in D, for every $(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n) \in \mathscr{A}_q^{n-1}$ and every $i = 1,\dots,n$. Therefore we can identify the space $SY(A,D;Z)$ with the set $S(A,D,q)$ of all families $\boldsymbol{b} = (\boldsymbol{b}^{t_1\dots t_n})_{t_1<\dots<t_n}$, with $t_1 < \cdots < t_n$ varying in A, such that $\boldsymbol{b}^{t_1\dots t_n} = (\boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n})_{(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_q^n}$ satisfy properties (2.3.3) and (2.3.5).

Let $A = [0,T]$. Using the previous identification, we can rewrite the H-variation of a compatible system $\mu \in SY(A,D;Z)$ in the interval $(c,d) \subseteq [0,T]$ (see (2.1.8)) in terms of the family $\boldsymbol{b}$ corresponding to $\boldsymbol{\mu}$: $\mathrm{Var}_H(\boldsymbol{\mu}; c, d) = \mathrm{Diss}_H(\boldsymbol{b}; c, d)$, where

$$\mathrm{Diss}_H(\boldsymbol{b}; c, d) := \sup \sum_{i=1}^{k} \sum_{\alpha\beta} H(\theta_\beta, \theta_\alpha) \int_D b_{\alpha\beta}^{t_{i-1}t_i}(x)\, dx, \tag{2.3.6}$$

where the supremum is taken over all finite partitions $c = t_0 < \cdots < t_k = d$ of the interval $[c, d]$ (with the convention $\mathrm{Diss}_H(\boldsymbol{b}; c, d) = 0$, if $c = d$).

It is easy to see that given a sequence $(\mu^k)_k = (\sum_{\alpha=1}^q b_\alpha^k \boldsymbol{\delta}_{\theta_\alpha})_k$ in $Y(D; Z)$, $\mu^k \rightharpoonup \mu = \sum_{\alpha=1}^q b_\alpha \boldsymbol{\delta}_{\theta_\alpha}$ weakly* in $Y(D; Z)$ if and only if $b_\alpha^k \rightharpoonup b_\alpha$ L^∞-weakly*, for every $\alpha = 1, \ldots, q$. Therefore a compatible system $\boldsymbol{\mu} \in SY([0, T], D; Z)$ is left continuous if and only if the correspondent $\boldsymbol{b} \in S([0, T], D, q)$ satisfies the following property: for every finite sequence $t_1 < \cdots < t_n$ in $[0, T]$

$$\boldsymbol{b}_{\alpha_1\ldots\alpha_n}^{s_1\ldots s_n} \rightharpoonup \boldsymbol{b}_{\alpha_1\ldots\alpha_n}^{t_1\ldots t_n} \quad L^\infty\text{-weakly*}, \tag{2.3.7}$$

as $s_i \to t_i$, with $s_i \in [0, T]$ and $s_i \leq t_i$, for every $(\alpha_1, \ldots, \alpha_n) \in \mathscr{A}_q^n$. We denote the set of all $\boldsymbol{b} \in S([0, T], D, q)$ satisfying (2.3.7) by $S_-([0, T], D, q)$.

Definition 2.3.1. Fixed a sequence $0 = t_1 < \cdots < t_m = T$ in $[0, T]$, and given a family $(b_{\alpha_1\ldots\alpha_m})_{(\alpha_1,\ldots,\alpha_m)\in\mathscr{A}_q^m}$ in $L^\infty(D; [0, 1])$ with $\sum_{(\alpha_1,\ldots,\alpha_m)} b_{\alpha_1\ldots\alpha_m} = 1$ a.e. in D, we define $\boldsymbol{b}^{pwc} \in S([0, T], D, q)$ as the family corresponding to the piecewise constant interpolation of the measure $\mu = \sum_{(\alpha_1\ldots\alpha_m)} b_{\alpha_1\ldots\alpha_m} \boldsymbol{\delta}_{(\theta_{\alpha_1},\ldots,\theta_{\alpha_m})}$, as defined in (2.1.8).

More in details, for every finite sequence $\tau_1 < \cdots < \tau_n$ in $[0, T]$ such that for every $j = 1, \ldots, m$ there exists $i = 1, \ldots, n$ with $t_j \leq \tau_i < t_{j+1}$, we have

$$(\boldsymbol{b}^{pwc})_{\beta_1\ldots\beta_n}^{\tau_1\ldots\tau_n} := \begin{cases} 0 & \text{if } \beta_i \neq \beta_{i+1} \text{ with } t_j \leq \tau_i < \tau_{i+1} < t_{j+1} \\ & \text{for some } i \text{ and } j \\ b_{\alpha_1\ldots\alpha_m} & \text{otherwise,} \end{cases}$$

for every $(\beta_1, \ldots, \beta_n) \in \mathscr{A}_q^n$.

We can reformulate Helly's Theorem for compatible systems of Young measures (see Theorem 2.1.6) in the discrete setting as follows.

Theorem 2.3.2. *Let $(\boldsymbol{b}^k)_k$ be a sequence in $S([0, T], D, q)$ such that $\mathrm{Diss}_H(\boldsymbol{b}^k; 0, T) \leq C$, for a finite constant $C > 0$ independent of k. Then there exist a subsequence, still denoted by $(\boldsymbol{b}^k)_k$, a set $\mathcal{T} \subseteq [0, T]$ containing 0 and at most countable, and $\boldsymbol{b} \in S_-([0, T], D, q)$ with $\mathrm{Diss}_H(\boldsymbol{b}; 0, T) \leq C$, such that*

$$(\boldsymbol{b}^k)_{\alpha_1\ldots\alpha_n}^{t_1\ldots t_n} \rightharpoonup \boldsymbol{b}_{\alpha_1\ldots\alpha_n}^{t_1\ldots t_n} \quad L^\infty\text{-weakly*}, \tag{2.3.8}$$

as $k \to \infty$, for every finite sequence $t_1 < \cdots < t_n$ in $\mathcal{T}$, and every $(\alpha_1, \ldots, \alpha_n) \in \mathscr{A}_q^n$.

Now we state a lemma to describe the canonical form of the space $Y^p(D; Z \times \mathbb{R}^{N\times d})$ of all Young measures on D with values in $Z \times \mathbb{R}^{N\times d}$ and finite p-moments, for $2 \leq p < +\infty$.

Lemma 2.3.3. *A measure ν is an element of $Y^p(D; Z \times \mathbb{R}^{N\times d})$ if and only if it can be written as*

$$\nu = \sum_{\alpha=1}^{q} b_\alpha (\delta_{\theta_\alpha} \otimes \lambda_\alpha), \tag{2.3.9}$$

where $(b_\alpha)_{\alpha=1}^q$ *is a family in* $L^\infty(D;[0,1])$ *satisfying (2.3.2) and, for every* $\alpha = 1,\ldots,q$, λ_α *is a Young measure on* D *with values in* $\mathbb{R}^{N\times d}$ *such that*

$$\int_{D\times\mathbb{R}^{N\times d}} b_\alpha(x)|F|^p\, d\lambda_\alpha(x,F) < \infty, \tag{2.3.10}$$

for every $\alpha = 1,\ldots,q$.

Proof. For every $\alpha = 1,\ldots,q$, let us consider $b_\alpha \in L^\infty(D;[0,1])$ and a Young measure $\lambda_\alpha \in Y(D;\mathbb{R}^{N\times d})$, satisfying (2.3.2) and (2.3.10). It is immediate to see that the measure defined by (2.3.9) is an element of $Y^p(D;Z\times\mathbb{R}^{N\times d})$.

On the other hand, if ν belongs to $\in Y^p(D;Z\times\mathbb{R}^{N\times d})$ and $(\nu^x)_{x\in D}$ is its disintegration, for a.e. $x\in D$ and every $\alpha = 1,\ldots,q$ we define

$$b_\alpha(x) := \nu^x(\{\theta_\alpha\}\times\mathbb{R}^{N\times d}); \tag{2.3.11}$$

let us fix a probability measure ω on $\mathbb{R}^{N\times d}$; for $\alpha = 1,\ldots,q$ and for a.e. $x\in D$ let us define a probability measure λ_α^x on $\mathbb{R}^{N\times d}$ by setting for every $B_\alpha \in \mathcal{B}(\mathbb{R}^{N\times d})$

$$\lambda_\alpha^x(B_\alpha) := \begin{cases} \frac{\nu^x(\{\theta_\alpha\}\times B_\alpha)}{b_\alpha(x)} & \text{if } b_\alpha(x)\neq 0 \\ \omega(B_\alpha) & \text{if } b_\alpha(x) = 0 \end{cases} \tag{2.3.12}$$

By construction b_α is measurable with nonnegative values for every α, $\sum_{\alpha=1}^q b_\alpha(x) = \nu^x(Z\times\mathbb{R}^{N\times d}) = 1$ for a.e. $x\in D$, and $(\lambda_\alpha^x)_x$ is a measurable family of probability measures satisfying (2.3.10), for every α . It is now immediate to see that the measure $\tilde\nu$ whose disintegration is given by

$$\tilde\nu^x = \sum_{\alpha=1}^q b_\alpha(x)(\delta_{\theta_\alpha}\otimes\lambda_\alpha^x) \quad \text{for a.e. } x\in D, \tag{2.3.13}$$

is exactly ν. Indeed every Borel subset B of $Z\times\mathbb{R}^{N\times d}$ can be written as the union of disjoint sets of the form $\{\theta_\alpha\}\times B_\alpha$, for suitable $B_\alpha\in\mathcal{B}(\mathbb{R}^{N\times d})$, for $\alpha = 1,\ldots,q$; hence we have

$$\tilde\nu^x(B) = \sum_{\alpha=1}^q b_\alpha(x)\lambda_\alpha^x(B_\alpha) =$$
$$= \sum_{\alpha=1}^q b_\alpha(x)\frac{\nu^x(\{\theta_\alpha\}\times B_\alpha)}{b_\alpha(x)} = \nu^x(B).$$

$\square$

Remark 2.3.4. The functions b_α and the measures $b_\alpha\lambda_\alpha$ satisfying the properties described in the previous lemma are uniquely determined by ν. In particular if we consider the disintegration of λ_α, $(\lambda_\alpha^x)_{x\in D}$, we obtain that λ_α^x is uniquely determined for a.e. x in $\{x\in D\ :\ b_\alpha(x) > 0\}$.

Remark 2.3.5. Let $\nu^k = \sum_\alpha b_\alpha^k(\delta_{\theta_\alpha}\otimes\lambda_\alpha^k)$, $\nu = \sum_\alpha b_\alpha(\delta_{\theta_\alpha}\otimes\lambda_\alpha)$ belong to $Y^p(D;Z\times\mathbb{R}^{N\times d})$. A simple computation shows that a sequence $(\nu^k)_k$ in $Y^p(D;Z\times\mathbb{R}^{N\times d})$ p-weakly* converges to $\nu\in Y^p(D;Z\times\mathbb{R}^{N\times d})$ if and only if

$$b_\alpha^k\lambda_\alpha^k \rightharpoonup b_\alpha\lambda_\alpha \quad p\text{-weakly*} \tag{2.3.14}$$

for every $\alpha = 1,\ldots,q$.

We can rewrite Remark 1.3.4 in terms of $(b_\alpha, \lambda_\alpha)_\alpha$ as follows.

REMARK 2.3.6. For every $\alpha = 1, \ldots, q$, let $(b^h_\alpha, \lambda^h_\alpha)_h$ be a sequence in $L^\infty(D; [0,1]) \times Y(D; \mathbb{R}^{N\times d})$, satisfying (2.3.2) for every h, and

$$\sup_h \int_{D\times\mathbb{R}^{N\times d}} b^h_\alpha(x)|F|^p \, d\lambda^h_\alpha(x, F) \le C,$$

for every α, for a suitable positive constant C. Then there exists $(b_\alpha, \lambda_\alpha) \in L^\infty(D; [0,1]) \times Y(D; \mathbb{R}^{N\times d})$, for $\alpha = 1, \ldots, q$, satisfying (2.3.2) and (2.3.10), and such that, up to a subsequence,

$$\begin{gathered} b^h_\alpha \rightharpoonup b_\alpha \quad L^\infty\text{-weakly*} \\ b^h_\alpha \lambda^h_\alpha \rightharpoonup b_\alpha \lambda_\alpha \quad p\text{-weakly*}, \end{gathered}$$

as $h \to \infty$.

CHAPTER 3

Globally stable quasistatic evolution

3.1. Introduction

In this chapter we propose an approach to the quasistatic evolution problem considered in [**21**], which does not require any assumption on the convexity/quasiconvexity of the energy functional and does not need any regularizing term.

We adapt the standard construction of approximate solutions via time-discretized minimum problems to the more general setting of Young measures.

Given an initial value of the variables (z_0, v_0) and a partition of the time interval $[0,T]$ in which we study the evolution

$$0 = t^0 < t^1 < \cdots < t^k = T,$$

the approximate solution should be defined inductively by solving the following incremental minimum problem:

$$\inf \{\mathcal{W}(z,v) - \langle \boldsymbol{l}(t^i), v\rangle + \mathcal{H}(z - z(t^{i-1}))\} \tag{3.1.1}$$

among all (z,v) which make the energy finite and satisfy the boundary condition at time t^i. As anticipated in the introduction, these problems are not well-posed in Sobolev spaces and we will present an explicit example in which (3.1.1) has actually no solution (see Remark 3.4.3).

To obtain lower semicontinuity and coercivness of the energy functional we place the problem in a suitable space of Young measures and solve the incremental minimum problems in this extended setting.

The next step is the study of the convergence of the approximate solutions as the time step $t^i - t^{i-1}$ tends to 0. Up to careful choices of subsequences, we can obtain the convergence of the approximations to a pair $(\boldsymbol{\nu}, \boldsymbol{\mu})$, with $\boldsymbol{\nu}$ a time-dependent family of Young measures with finite second moments and values in $\mathbb{R}^m \times \mathbb{R}^{N\times d}$, and $\boldsymbol{\mu}$ a compatible system of Young measures with finite second moments and values in $\mathbb{R}^m$, connected to $\boldsymbol{\nu}$ by a suitable projection property.

The main result is Theorem 3.3.15, which shows that this pair satisfies a global stability condition and an energy inequality, suitably reformulated in Young measure language (see Definition 3.3.14); therefore it can be considered as a solution of the quasistatic evolution problem in the framework of Young measures.

This result is formulated in terms of stochastic processes too. Thanks to Theorem 2.2.4, the pair $(\boldsymbol{\nu}, \boldsymbol{\mu})$ representing the solution can be described by a unique stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ with values in $\mathbb{R}^m \times \mathbb{R}^{N\times d}$.

In the last section we give an alternative proof of the main theorem under the special assumption that the stored energy density is quasiconvex with respect to its second argument; we use the result in [**21**, Section 4] to obtain solutions of spatially regularized problems, and prove that we can pass to the limit as the regularization parameter vanishes, to obtain a globally stable quasistatic evolution in terms of Young measures in the sense of Definition 3.3.14.

3.2. The mechanical model

The *reference configuration* D is a bounded connected open subset of $\mathbb{R}^d$ with Lipschitz boundary $\partial D = \Gamma_0 \cup \Gamma_1$, where Γ_0 is assumed to be a nonempty closed subset of ∂D with $\mathcal{H}^{d-1}(\Gamma_0) \neq 0$, and $\Gamma_1 = \partial D \setminus \Gamma_0$. Without loss of generality, we also assume for simplicity that

$$\mathcal{L}^d(D) = 1. \tag{3.2.1}$$

We will indicate the *deformation* by v and the *internal variable* by z. We will denote *the stored energy density* by $W\colon \mathbb{R}^m \times \mathbb{R}^{N\times d} \to [0,+\infty)$ and the *dissipation rate density* by $H\colon \mathbb{R}^m \to [0,+\infty)$. For every $\theta, \tilde{\theta} \in \mathbb{R}^m$ and $F \in \mathbb{R}^{N\times d}$, we will make the following assumptions:

(W.1) there exist positive constants c, C such that

$$c(|\theta|^2 + |F|^2) - C \leq W(\theta, F) \leq C(1 + |\theta|^2 + |F|^2);$$

(W.2) $W(\theta,\cdot)$ is of class $\mathcal{C}^2$,

$$\Big|\frac{\partial W}{\partial F}(\theta, F)\Big| \leq C(1 + |\theta| + |F|),$$

and

$$|W(\theta + \tilde{\theta}, F) - W(\theta, F)| \leq C|\tilde{\theta}|(1 + |\theta| + |\tilde{\theta}| + |F|);$$

(H.1) H is positively homogeneous of degree one and convex;

(H.2) there exists a positive constant λ, such that $\frac{1}{\lambda}|\theta| \leq H(\theta) \leq \lambda|\theta|$.

Let $\mathcal{W}$ be the functional $\mathcal{W}(z,v) := \int_D W(z(x), \nabla v(x))\, dx$, for every $z \in L^2(D;\mathbb{R}^m)$ and every $v \in H^1(D;\mathbb{R}^N)$, and $\mathcal{H}$ the functional $\mathcal{H}(z) := \int_D H(z(x))\, dx$, for every $z \in L^1(D;\mathbb{R}^m)$.

Given two distinct times $s < t$, the *global dissipation* of a possibly discontinuous function $\boldsymbol{z}\colon [0,T] \to L^2(D;\mathbb{R}^m)$ in the interval $[s,t]$ will be

$$\mathrm{Var}_H(\boldsymbol{z}; s, t) := \sup \sum_{i=1}^{k} \mathcal{H}(\boldsymbol{z}(\tau_i) - \boldsymbol{z}(\tau_{i-1})),$$

where the supremum will be taken among all finite partitions $s = \tau_0 < \tau_1 < \cdots < \tau_k = t$.

The *external load* at time t and the prescribed *boundary condition* on Γ_0 at time t are denoted by $\boldsymbol{l}(t)$ and $\boldsymbol{\varphi}(t)$, respectively; we assume $\boldsymbol{l} \in AC([0,T]; H^1(D;\mathbb{R}^N)^*)$ and $\boldsymbol{\varphi} \in AC([0,T]; H^1(D;\mathbb{R}^N))$.

The kinematically admissible values at time t for z and v are those which make the total energy finite and satisfy the boundary condition, i.e., $v = \boldsymbol{\varphi}(t)$ on Γ_0 $\mathcal{H}^{d-1}$-a.e. (in the sense of traces). From the previous assumption it follows that the kinematically admissible values at time t are contained in $L^2(D;\mathbb{R}^m) \times \mathcal{A}(t)$, where

$$\mathcal{A}(t) = H^1_{\Gamma_0}(\boldsymbol{\varphi}(t)) := \{v \in H^1(D;\mathbb{R}^N) : v = \boldsymbol{\varphi}(t) \ \ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_0\}.$$

3.3. Globally stable quasistatic evolution for Young measures

3.3.1. Admissible set in terms of stochastic processes. Now we describe the set of admissible stochastic processes in which we look for a solution of our quasistatic evolution problem: the definition takes into account approximation properties with functions which satisfy the boundary condition.

DEFINITION 3.3.1. Given $A \subset \mathbb{R}$ and $\boldsymbol{w} \colon A \to H^1(D;\mathbb{R}^N)$, we define $AY_{sp}(A, \boldsymbol{w})$ as the set of all stochastic processes $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in A}$ on a $(D, \mathcal{L}^d)$-probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$ with

$$\boldsymbol{Z}_t \in L^2(D \times \Omega; \mathbb{R}^m),$$
$$\boldsymbol{Y}_t \in L^2(D \times \Omega; \mathbb{R}^{N\times d}),$$

satisfying the following property: for every finite sequence $t_1 < \cdots < t_n$ in A there exist sequences $(z_i^k)_k \subset L^2(D;\mathbb{R}^m)$, $(v_i^k)_k \subset H^1_{\Gamma_0}(\boldsymbol{w}(t_i))$, for $i = 1, \dots, n$ such that

(ap$_{\mathrm{sp}}$1) we have

$$(\pi_D, z_1^k, \dots, z_n^k)(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P)$$

2-weakly* as $k \to \infty$;

(ap$_{\mathrm{sp}}$2) for every $i = 1, \dots, n$, there exist a subsequence, possibly depending on i, $(z_i^{k_j^i}, v_i^{k_j^i})_j$, such that

$$(\pi_D, z^{k_j^i}, \nabla v^{k_j^i})(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_{t_i}, \boldsymbol{Y}_{t_i})(P)$$

2-weakly* as $j \to \infty$.

3.3.2. Admissible set in terms of Young measures. The notion of admissible set is now presented in terms of Young measures.

DEFINITION 3.3.2. Given $A \subset \mathbb{R}$ and $\boldsymbol{w} \colon A \to H^1(D;\mathbb{R}^N)$, we define $AY(A, \boldsymbol{w})$ as the set of all pairs $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D;\mathbb{R}^m \times \mathbb{R}^{N\times d})^A \times SY^2(A, D;\mathbb{R}^m)$ satisfying the following property: for every finite sequence $t_1 < \cdots < t_n$ in A there exist sequences $(z_i^k)_k \subset L^2(D;\mathbb{R}^m)$, $(v_i^k)_k \subset H^1_{\Gamma_0}(\boldsymbol{w}(t_i))$, for $i = 1, \dots, n$ such that

(ap1) we have

$$\boldsymbol{\delta}_{(z_1^k,\dots,z_n^k)} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_n}, \tag{3.3.1}$$

2-weakly* as $k \to \infty$;

(ap2) for every $i = 1, \dots, n$, there exists a subsequence, possibly depending on i, $(z_i^{k_j^i}, v_i^{k_j^i})_j$, such that

$$\boldsymbol{\delta}_{(z_i^{k_j^i}, \nabla v_i^{k_j^i})} \rightharpoonup \boldsymbol{\nu}_{t_i} \tag{3.3.2}$$

2-weakly* as $j \to \infty$.

REMARK 3.3.3. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$, then $\tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$, for every $t \in A$. Indeed, fixed $t \in A$, by definition there exist $(z^k)_k \subset L^2(D;\mathbb{R}^m)$, $(v^k)_k \subset H^1_{\Gamma_0}(\boldsymbol{w}(t))$ such that $\boldsymbol{\delta}_{(z^k,\nabla v^k)} \rightharpoonup \boldsymbol{\nu}_t$ 2-weakly* and $\boldsymbol{\delta}_{z^k} \rightharpoonup \boldsymbol{\mu}_t$ 2-weakly*; in particular $\tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\delta}_{(z^k,\nabla v^k)}) \rightharpoonup \tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\nu}_t)$ 2-weakly* and this prove the claim.

REMARK 3.3.4. If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in A} \in AY_{sp}(A, \boldsymbol{w})$, we can define $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$ as

$$\boldsymbol{\nu}_t := (\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) \quad \text{for every } t \in A$$
$$\boldsymbol{\mu}_{t_1\dots t_n} := (\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P) \quad \text{for every finite sequence } t_1 < \dots < t_n \text{ in } A.$$

On the other side, thanks to Remark 3.3.3 and Theorem 2.2.4, for every $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$ there exists a stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in A} \in AY_{sp}(A, \boldsymbol{w})$ such that, for every finite sequence $t_1 < \dots < t_n$ in A,

$$(\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P) = \boldsymbol{\mu}_{t_1\dots t_n}$$
$$(\pi_D, \boldsymbol{Z}_{t_i}, \boldsymbol{Y}_{t_i})(P) = \boldsymbol{\nu}_{t_i} \quad \text{for every } i = 1, \dots, n.$$

REMARK 3.3.5. Thanks to Decomposition Lemma 1.3.8 and Lemma 1.3.9, given $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$ and a finite sequence $t_1 < \dots < t_m$ in A, we can always choose $z_i^k \in L^2(D; \mathbb{R}^m)$ and $v_i^k \in H^1_{\Gamma_0}(\boldsymbol{w}(t^i))$, for $i = 1, \dots, m$, in such a way that $|z_i^k|^2$ are equiintegrable, satisfy (3.3.1), and for every i there exists a subsequence $(z_i^{k_j^i}, v_i^{k_j^i})_j$ satisfying (3.3.2), such that $|\nabla v_i^{k_j^i}|^2$ are equiintegrable with respect to j. Hence, by Theorem 1.3.10, we can always assume that

$$\|(z_1^k, \dots z_n^k)\|_2^2 \to \int_{D\times(\mathbb{R}^m)^n} |(\theta_1, \dots, \theta_n)|^2 \, d\boldsymbol{\mu}_{t_1\dots t_n}(x, \theta_1, \dots, \theta_n), \tag{3.3.3}$$

$$\|(z_i^{k_j^i}, \nabla v_i^{k_j^i})\|_2^2 \to \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} |(\theta, F)|^2 \, d\boldsymbol{\nu}_{t_i}(x, \theta, \xi), \tag{3.3.4}$$

as $j \to \infty$, for $i = 1, \dots, m$. This allows us to assume, without loss of generality, that

$$\sup_k \|z_i^k\|_2^2 \le C_1 + 1, \tag{3.3.5}$$

$$\sup_j \|(z_i^{k_j^i}, \nabla v_i^{k_j^i})\|_2^2 \le C_2 + 1, \tag{3.3.6}$$

with

$$C_1 := \sup_{i=1,\dots,n} \int_{D\times\mathbb{R}^m} |\theta|^2 \, d\boldsymbol{\mu}_{t_i}(x, \theta),$$
$$C_2 := \sup_{i=1,\dots,n} \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} |(\theta, F)|^2 \, d\boldsymbol{\nu}_{t_i}(x, \theta, F).$$

In the following two Lemmas we want to point out some closure properties of $AY(A, \boldsymbol{w})$.

LEMMA 3.3.6. *Let $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^A \times SY^2(A, D; \mathbb{R}^m)$, and assume that for every finite sequence $t_1 < \dots < t_n$ in A there exists a sequence $(\boldsymbol{\nu}^j, \boldsymbol{\mu}^j)_j$ in $AY(\{t_1, \dots, t_n\}, \boldsymbol{w})$, such that*

$$\boldsymbol{\mu}^j_{t_1\dots t_n} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_n} \quad \text{2-weakly*}, \tag{3.3.7}$$

as $j \to \infty$, and such that for every i there exists a subsequence, possibly depending on i, $(\boldsymbol{\nu}^{j_h^i})_h$, satisfying

$$(\boldsymbol{\nu}^{j_h^i})_{t_i} \rightharpoonup \boldsymbol{\nu}_{t_i}, \quad \text{2-weakly*}, \tag{3.3.8}$$

as $h \to \infty$. Then $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$.

PROOF. Fix a finite sequence $t_1 < \cdots < t_n$ in A. By definition of $AY(\{t_1, \ldots, t_n\}, \boldsymbol{w})$, for every j and every $i = 1, \ldots, n$, there exist $(z_i^{j,k})_k \in L^2(D; \mathbb{R}^m)$ and $(v_i^{j,k})_k \in H^1_{\Gamma_0}(\boldsymbol{w}(t_i))$ satisfying (3.3.1) for $\boldsymbol{\mu}_j$ and such that for every i and j there exists an increasing sequence of integers $(k_l^{i,j})_l$ for which $(z_i^{j,k_l^{i,j}}, v_i^{j,k_l^{i,j}})_l$ satisfies (3.3.2) for $\boldsymbol{\nu}^j$; thanks to Remark 3.3.5 we can assume, without loss of generality, that

$$\|z_i^{j,k}\|_2^2 \leq \int_{D\times\mathbb{R}^m} |\theta|^2 \, d\boldsymbol{\mu}^j_{t_i}(x,\theta) + 1, \quad \text{for every } \ k,$$

and

$$\|\nabla v_i^{j,k_l^{i,j}}\|_2^2 \leq \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} |(\theta, F)|^2 \, d\boldsymbol{\nu}^j_{t_i}(x,\theta,F) + 1, \quad \text{for every } \ l;$$

hence thanks to (3.3.7) and (3.3.8) there exists a positive constant C such that

$$\|z_i^{j,k}\|_2^2 \leq C + 1, \tag{3.3.9}$$

for every i, j, k, and

$$\sup_h \sup_l \|\nabla v_i^{j_h^i, k_l^{i,j_h^i}}\|_2^2 \leq C + 1, . \tag{3.3.10}$$

for every i, h, l. Thanks to Remark 1.3.2, we can find a metric d_1 on $Y(D; (\mathbb{R}^m)^n)$ and a metric d_2 on $Y(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})$ which induce the weak* topologies of $Y(D; (\mathbb{R}^m)^n)$ and $Y(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})$, respectively; therefore, for every j we can find an integer $\kappa(j)$ such that, for every $k \geq \kappa(j)$ it holds

$$d_1(\boldsymbol{\delta}_{(z_1^{j,k},\ldots,z_n^{j,k})}, \boldsymbol{\mu}^j_{t_1\ldots t_n}) < \frac{1}{j}; \tag{3.3.11}$$

analogously, for every $i = 1, \ldots, n$, there exists an integer $\kappa_i(j)$ such that,

$$d_2(\boldsymbol{\delta}_{(z_i^{j,k_l^{i,j}}, \nabla v_i^{j,k_l^{i,j}})}, \boldsymbol{\nu}^j_{t_i}) < \frac{1}{j} \tag{3.3.12}$$

whenever $k_l^{i,j} \geq \kappa_i(j)$.

By taking, if needed, a larger value of $\kappa(j)$, we may assume that (3.3.11) and (3.3.12) are satisfied whenever $k \geq \kappa(j)$ and $k_l^{i,j} \geq \kappa(j)$, respectively. Another slight modification allows us to assume that, for every $i = 1, \ldots, n$ and for every j, there exists $k_{l_{i,j}}^{i,j}$ with

$$\kappa(j) < k_{l_{i,j}}^{i,j} \leq \kappa(j+1). \tag{3.3.13}$$

Let $(\alpha(k))_{k>\kappa(1)}$ be the unique sequence such that $\kappa(\alpha(k)) < k \leq \kappa(\alpha(k)+1)$, for every $k > \kappa(1)$. This implies

$$d_1(\boldsymbol{\delta}_{(z_1^{\alpha(k),k},\ldots,z_n^{\alpha(k),k})}, \boldsymbol{\mu}^{\alpha(k)}_{t_1\ldots t_n}) \leq \frac{1}{\alpha(k)}, \tag{3.3.14}$$

which, together with (3.3.7) and (3.3.9), implies that

$$\boldsymbol{\delta}_{(z_1^{\alpha(k),k},\ldots,z_n^{\alpha(k),k})} \rightharpoonup \boldsymbol{\mu}_{t_1\ldots t_n}, \tag{3.3.15}$$

2-weakly* as $k \to \infty$.

Now, for every $i = 1, \dots, n$ we can choose an integer $\beta_i(j)$ in such a way that

$$\beta_i(j) = k^{i,j}_{l_{i,j}}, \tag{3.3.16}$$

for every j, so that we have $\kappa(j) < \beta_i(j) \leq \kappa(j+1)$, for every j, by(3.3.13). This implies that $\alpha(\beta_i(j)) = j$ and $\beta_i(j) > \kappa(j)$ so that, by (3.3.12)

$$d_2(\boldsymbol{\delta}_{(z_i^{j,\beta_i(j)}, \nabla v_i^{j,\beta_i(j)})}, \boldsymbol{\nu}^j_{t_i}) < \frac{1}{j}.$$

Therefore, thanks to (3.3.8) and (3.3.10) we can conclude that

$$\boldsymbol{\delta}_{(z_i^{\alpha(\beta_i(j^i_h)),\beta_i(j^i_h)}, \nabla v_i^{\alpha(\beta_i(j^i_h)),\beta_i(j^i_h)})} \rightharpoonup \boldsymbol{\nu}_{t_i}, \tag{3.3.17}$$

as $h \to \infty$, for every $i = 1, \dots, n$. Since for every i the sequence $(\beta_i(j^i_h))_h$ is increasing, (3.3.15) and (3.3.17) show that conditions (ap1) and (ap2) in Definition 3.3.2 are satisfied. □

The following lemma consider the case of varying boundary conditions.

LEMMA 3.3.7. *Let $\boldsymbol{w}^j$ be a sequence of functions from A into $H^1(D, \mathbb{R}^m)$, such that $\boldsymbol{w}^j(t) \to \boldsymbol{w}(t)$ strongly in H^1, for every $t \in A$ and let $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^A \times SY^2(A, D; \mathbb{R}^m)$. Assume that for every finite sequence $t_1 < \cdots < t_n$ in A there exists a sequence $(\boldsymbol{\nu}^j, \boldsymbol{\mu}^j) \in AY(\{t_1, \dots, t_n\}, \boldsymbol{w}^j)$ such that*

$$\boldsymbol{\mu}^j_{t_1 \dots t_n} \rightharpoonup \boldsymbol{\mu}_{t_1 \dots t_n} \quad \text{2-weakly*}, \tag{3.3.18}$$

as $j \to \infty$, and such that for every i there exists a subsequence, possibly depending on i, $(\boldsymbol{\nu}^{j^i_h})_h$, satisfying

$$(\boldsymbol{\nu}^{j^i_h})_{t_i} \rightharpoonup \boldsymbol{\nu}_{t_i}, \quad \text{2-weakly*}, \tag{3.3.19}$$

as $h \to \infty$. Then $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$.

PROOF. Fixed $t_1 < \cdots < t_n$ in $[0, T]$, thanks to Lemma 1.3.7 from (3.3.19) we can deduce that for every $i = 1, \dots, n$

$$\tilde{\mathcal{T}}^2_{\nabla \boldsymbol{w}(t_i) - \nabla \boldsymbol{w}^{j^i_h}(t_i)} (\boldsymbol{\nu}^{j^i_h})_{t_i} \rightharpoonup \boldsymbol{\nu}_{t_i} \quad \text{2-weakly*}$$

as $h \to \infty$, where $\tilde{\mathcal{T}}^2_{\nabla \boldsymbol{w}(t_i) - \nabla \boldsymbol{w}^{j^i_h}(t_i)}$ is the map defined by $\tilde{\mathcal{T}}^2_{\nabla \boldsymbol{w}(t_i) - \nabla \boldsymbol{w}^{j^i_h}(t_i)}(x, \theta, F) := (x, \theta, F + \nabla \boldsymbol{w}(t_i) - \nabla \boldsymbol{w}^{j^i_h}(t_i))$. Thanks to Lemma 1.3.6 it is easy to see that the hypotheses of Lemma 3.3.6 are satisfied with $(\boldsymbol{\nu}^{j^i_h})_{t_i}$ replaced by $\tilde{\mathcal{T}}^2_{\nabla \boldsymbol{w}(t_i) - \nabla \boldsymbol{w}^{j^i_h}(t_i)}((\boldsymbol{\nu}^{j^i_h})_{t_i})$. □

REMARK 3.3.8. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$, for every $t \in A$ there exists a unique function $\boldsymbol{v}(t) \in H^1_{\Gamma_0}(\boldsymbol{w}(t))$ such that $\nabla \boldsymbol{v}(t) = \text{bar}(\tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$. Indeed, by definition of $AY(A, \boldsymbol{w})$, for every $t \in A$ there exists a sequence $v^k \in H^1_{\Gamma_0}(\boldsymbol{w}(t))$ such that $\boldsymbol{\delta}_{\nabla v^k} \rightharpoonup \tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t)$, 2-weakly*; thanks to a variant of Lemma 1.3.11 with H^1_0 replaced by $H^1_{\Gamma_0}(\boldsymbol{w}(t))$, there exists a unique function $\boldsymbol{v}(t) \in H^1(D; \mathbb{R})$, such that $v^k \rightharpoonup \boldsymbol{v}(t)$ weakly in H^1 and $\nabla \boldsymbol{v}(t) = \text{bar}(\tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$

Translating the previous remark in terms of stochastic processes we obtain the following result.

REMARK 3.3.9. If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]} \in AY_{sp}([0,T], \boldsymbol{\varphi})$, for every $t \in A$ there exists a unique function $\boldsymbol{v}(t) \in H^1_{\Gamma_0}(\boldsymbol{w}(t))$ such that $\nabla \boldsymbol{v}(t) = \mathrm{bar}((\pi_D, \boldsymbol{Y}_t)(P))$.

REMARK 3.3.10. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(A, \boldsymbol{w})$, for every $t \in A$ we define

$$\boldsymbol{\sigma}(t,x) := \int_{\mathbb{R}^m \times \mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta, F)\; d\boldsymbol{\nu}^x_t(\theta, F), \tag{3.3.20}$$

for a.e. $x \in D$. For every $t \in A$ we have that $\boldsymbol{\sigma}(t) \in L^2(D; \mathbb{R}^{N\times d})$: this comes immediately from **(W.2)**, (1.3.2), and from the fact that $\boldsymbol{\nu}_t \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})$. In the language of stochastic processes $\boldsymbol{\sigma}(t)$ can be characterized as the unique element of $L^2(D; \mathbb{R}^{N\times d})$ such that

$$\int_D \boldsymbol{\sigma}(t,x) g(x)\; dx = \int_{D\times\Omega} \frac{\partial W}{\partial F}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) g(x)\; dP(x,\omega), \tag{3.3.21}$$

for every $g \in L^2(D; \mathbb{R}^{N\times d})$, where $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is the stochastic process corresponding to $(\boldsymbol{\nu}, \boldsymbol{\mu})$.

REMARK 3.3.11. Since $\boldsymbol{\varphi} \in AC([0,T]; H^1(D; \mathbb{R}^N))$ and $\boldsymbol{l} \in AC([0,T]; H^1(D; \mathbb{R}^N)^*)$, the time derivative $\dot{\boldsymbol{\varphi}}$ and $\dot{\boldsymbol{l}}$ are well defined for a.e. $t \in [0,T]$ and belong to the space $L^1([0,T]; H^1(D; \mathbb{R}^N))$ and $L^1([0,T]; H^1(D; \mathbb{R}^N)^*)$, respectively. Moreover the fundamental Theorem of Calculus holds (see, e.g., [**6**, Appendice]).

3.3.3. Main result. We are now in the position to define the notion of globally stable quasistatic evolution of stochastic processes.

DEFINITION 3.3.12. Given $\boldsymbol{\varphi} \in AC([0,T]; H^1(D; \mathbb{R}^N))$, $\boldsymbol{l} \in AC([0,T]; H^1(D; \mathbb{R}^N)^*)$, $z_0 \in L^2(D; \mathbb{R}^m)$, $v_0 \in \mathcal{A}(0)$, and $T > 0$, a *globally stable quasistatic evolution of stochastic processes* with boundary datum $\boldsymbol{\varphi}$, external load $\boldsymbol{l}$, and initial condition (z_0, v_0), in the time interval $[0,T]$, is a stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]} \in AY_{sp}([0,T], \boldsymbol{\varphi})$, such that for every finite sequence $t_1 < \cdots < t_n$ in $[0,T]$ we have

$$(\pi_D, \boldsymbol{Z}_{s^j_1}, \ldots, \boldsymbol{Z}_{s^j_n})(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_{t_1}, \ldots, \boldsymbol{Z}_{t_n})(P)$$

2-weakly*, as $s^j_i \to t_i$ with $s^j_i \le t_i$, and satisfying the following conditions:

(ev0) *initial condition*: $(\boldsymbol{Z}_0, \boldsymbol{Y}_0) = (z_0, \nabla v_0)$;

(ev1) *partial-global stability*: for every $t \in [0,T]$, we have

$$\int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega))\; dP(x,\omega) \le$$
$$\le \int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega) + \tilde{z}(x), \boldsymbol{Y}_t(x,\omega) + \nabla\tilde{u}(x))\; dP(x,\omega) - \langle \boldsymbol{l}(t), \tilde{u}\rangle + \mathcal{H}(\tilde{z}),$$

for every $\tilde{z} \in L^2(D; \mathbb{R}^m)$ and every $\tilde{u} \in H^1_{\Gamma_0}(0)$;

(ev2) *energy inequality*: for every $t \in [0,T]$ we have

$$\mathrm{Var}_H(\boldsymbol{Z}, P; 0, t) := \sup \sum_{i=1}^{k} \int_{D\times\Omega} H(\boldsymbol{Z}_{t_i}(x,\omega) - \boldsymbol{Z}_{t_{i-1}}(x,\omega))\; dP(x,\omega) < \infty, \tag{3.3.22}$$

where the supremum is taken over all finite partitions $0 = t_0 < \cdots < t_k = t$, and the map

$$t \mapsto [\langle \boldsymbol{\sigma}(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t) \rangle]$$

is integrable on $[0,T]$, where $\boldsymbol{\sigma}(t)$ is the function defined in (3.3.21) and $\boldsymbol{v}(t)$ that one defined in Remark 3.3.9; moreover

$$\int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega))\, dP(x,\omega) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{Z}, P; 0, t) \leq$$
$$\leq \mathcal{W}(z_0, v_0) - \langle \boldsymbol{l}(0), v_0 \rangle + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle\, ds +$$
$$- \int_0^t [\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle + \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}(s) \rangle]\, ds.$$

We now state the main existence theorem in terms of stochastic processes.

THEOREM 3.3.13. *Let $\boldsymbol{\varphi} \in AC([0,T]; H^1(D;\mathbb{R}^N))$, $\boldsymbol{l} \in AC([0,T]; H^1(D;\mathbb{R}^N)^*)$, $T > 0$, $z_0 \in L^2(D;\mathbb{R}^m)$ and $v_0 \in \mathcal{A}(0)$ be such that*

$$\mathcal{W}(z_0, v_0) \leq \mathcal{W}(z_0 + \tilde{z}, v_0 + \tilde{u}) - \langle \boldsymbol{l}(0), \tilde{u} \rangle + \mathcal{H}(\tilde{z}), \tag{3.3.23}$$

for every $\tilde{z} \in L^2(D;\mathbb{R}^m)$ and every $\tilde{u} \in H^1_{\Gamma_0}(0)$. Then there exists a globally stable quasistatic evolution for stochastic processes with boundary datum $\boldsymbol{\varphi}$, external load $\boldsymbol{l}$, and initial condition (z_0, v_0), in the time interval $[0,T]$.

Thanks to Remark 3.3.4, we can translate the definition of globally stable quasistatic evolution of stochastic processes in terms of globally stable quasistatic evolution of Young measures.

DEFINITION 3.3.14. Given $\boldsymbol{\varphi} \in AC([0,T]; H^1(D;\mathbb{R}^N))$, $\boldsymbol{l} \in AC([0,T]; H^1(D;\mathbb{R}^N)^*)$, $z_0 \in L^2(D;\mathbb{R}^m)$, $v_0 \in \mathcal{A}(0)$, and $T > 0$, a *globally stable quasistatic evolution of Young measures* with boundary datum $\boldsymbol{\varphi}$, external load $\boldsymbol{l}$ and initial condition (z_0, v_0), in the time interval $[0,T]$, is a pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY([0,T], \boldsymbol{\varphi})$, with $\boldsymbol{\mu} \in SY^2_-([0,T], D; \mathbb{R}^m)$, satisfying the following conditions:

(ev0) *initial condition*: $\boldsymbol{\nu}_0 = \boldsymbol{\delta}_{(z_0, \nabla v_0)}$;

(ev1) *partial-global stability*: for every $t \in [0,T]$, we have

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta, F)\, d\boldsymbol{\nu}_t(x,\theta,F) \leq$$
$$\leq \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta + \tilde{z}(x), F + \nabla\tilde{u}(x))\, d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \tilde{u} \rangle + \mathcal{H}(\tilde{z}),$$

for every $\tilde{z} \in L^2(D;\mathbb{R}^m)$ and every $\tilde{u} \in H^1_{\Gamma_0}(0)$;

(ev2) *energy inequality*: for every $t \in [0,T]$ we have that $\mathrm{Var}_H(\boldsymbol{\mu}; 0, t) < \infty$, (see (2.1.8)), and the map

$$t \mapsto [\langle \boldsymbol{\sigma}(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t) \rangle] \tag{3.3.24}$$

is integrable on $[0,T]$, where $\boldsymbol{\sigma}(t)$ is the function defined in (3.3.20) and $\boldsymbol{v}(t)$ that one defined in Remark 3.3.8; moreover

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\, d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t),\boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu};0,t) \leq$$
$$\leq \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0),v_0\rangle + \int_0^t \langle \boldsymbol{\sigma}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle\, ds +$$
$$-\int_0^t [\langle \boldsymbol{l}(s),\dot{\boldsymbol{\varphi}}(s)\rangle + \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}(s)\rangle]\, ds,$$

where $Var_H(\boldsymbol{\mu};0,t)$ is defined as in (2.1.8).

Thanks to Theorem 2.2.4, to obtain the main theorem it is enough to prove the following version for Young measures.

THEOREM 3.3.15. *Let $\boldsymbol{\varphi}\in AC([0,T];H^1(D;\mathbb{R}^N))$, $\boldsymbol{l}\in AC([0,T];H^1(D;\mathbb{R}^N)^*)$, $T>0$, $z_0\in L^2(D;\mathbb{R}^m)$, and $v_0\in\mathcal{A}(0)$ be such that*

$$\mathcal{W}(z_0,v_0)\leq \mathcal{W}(z_0+\tilde z,v_0+\tilde u) - \langle \boldsymbol{l}(0),\tilde u\rangle + \mathcal{H}(\tilde z), \tag{3.3.25}$$

for every $\tilde z\in L^2(D;\mathbb{R}^m)$ and every $\tilde u\in H^1_{\Gamma_0}(0)$. Then there exists a globally stable quasistatic evolution for Young measures with boundary datum $\boldsymbol{\varphi}$, external load $\boldsymbol{l}$, and initial condition (z_0,v_0), in the time interval $[0,T]$.

REMARK 3.3.16. In the proof of Theorem 3.3.15 we will obtain, in particular, a globally stable quasistatic evolution $(\boldsymbol{\nu},\boldsymbol{\mu})$ such that $\boldsymbol{\nu}$ is Θ-2-weakly* approximable from the left (see Definition 2.1.4), for a suitable subset Θ of $[0,T]$.

3.4. Proof of the main theorem

The proof is obtained via time discretization, resolution of incremental minimum problems, and passing to the limit as the discretization step tends to 0.

3.4.1. The incremental minimum problem. Let us fix a sequence of subdivisions of $[0,T]$, $0=t_n^0<t_n^1<\cdots<t_n^{k(n)}=T$, such that $\sup_{i=1,\dots,k(n)}\tau_n^i\to 0$, as $n\to\infty$, where $\tau_n^i:=t_n^i-t_n^{i-1}$, for every $i=1,\dots,k(n)$.

For every $i=0,1,\dots,k(n)$ we set $l_n^i:=\boldsymbol{l}(t_n^i)$ and $\varphi_n^i:=\boldsymbol{\varphi}(t_n^i)$.

We will define $(\boldsymbol{\nu}_n^i,\boldsymbol{\mu}_n^i)\in AY(\{t_n^0,\dots,t_n^i\},\boldsymbol{\varphi})$ by induction on i: set $(\boldsymbol{\nu}_n^0,\boldsymbol{\mu}_n^0):=\boldsymbol{\delta}_{(z_0,\nabla v_0)}$, and for $i>0$ we define $(\boldsymbol{\nu}_n^i,\boldsymbol{\mu}_n^i)$ as a minimizer (see Lemma 3.4.2 below) of the functional

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\, d\boldsymbol{\nu}_{t_n^i}(x,\theta,F) - \langle l_n^i,\boldsymbol{v}(t_n^i)\rangle +$$
$$+\textstyle\int_{D\times(\mathbb{R}^m)^2} H(\theta_i-\theta_{i-1})\, d\boldsymbol{\mu}_{t_n^{i-1}t_n^i}(x,\theta_{i-1},\theta_i), \tag{3.4.1}$$

in the set A_n^i of all $(\boldsymbol{\nu},\boldsymbol{\mu})\in AY(\{t_n^0,\dots,t_n^i\},\boldsymbol{\varphi})$, satisfying

$$\boldsymbol{\mu}_{t_n^0\dots t_n^{i-1}} = (\boldsymbol{\mu}_n^{i-1})_{t_n^0\dots t_n^{i-1}} \tag{3.4.2}$$
$$\boldsymbol{\nu}_{t_n^j} = (\boldsymbol{\nu}_n^{i-1})_{t_n^j},\quad \text{for every } j<i, \tag{3.4.3}$$

where the function $\boldsymbol{v}(t_n^i)$ appearing in (3.4.1) is that one defined in Remark 3.3.8.

LEMMA 3.4.1. *The set A_n^i is nonempty, for every $i > 1$.*

PROOF. Fixed $(\boldsymbol{\nu}_n^{i-1}, \boldsymbol{\mu}_n^{i-1})$, we consider the map

$$\tilde{\mathcal{T}}^2_{\nabla\varphi_n^i - \nabla\varphi_n^{i-1}} \colon (x, \theta, F) \mapsto (x, \theta, F + \nabla\varphi_n^i(x) - \nabla\varphi_n^{i-1}(x)),$$

and the map $pi^{(i)} \colon D \times (\mathbb{R}^m)^i \to D \times (\mathbb{R}^m)^{i+1}$ defined by

$$\pi^{(i)}(x, \theta_1, \ldots, \theta_{i-1}) := (x, \theta_1, \ldots, \theta_{i-1}, \theta_{i-1});$$

let $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m)^{\{t_n^0, \ldots, t_n^i\}}$ be defined by

$$\begin{aligned} \boldsymbol{\nu}_{t_n^j} &:= (\boldsymbol{\nu}_n^{i-1})_{t_n^j}, \quad \text{for } j < i, \\ \boldsymbol{\nu}_{t_n^i} &:= \tilde{\mathcal{T}}^2_{\nabla\varphi_n^i - \nabla\varphi_n^{i-1}}((\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}), \end{aligned} \tag{3.4.4}$$

and $\boldsymbol{\mu}$ the unique element of $SY^2(\{t_n^0, \ldots, t_n^i\}, D; \mathbb{R}^m)$ satisfying $\boldsymbol{\mu}_{t_n^0 \ldots t_n^i} = \pi^{(i)}((\boldsymbol{\mu}_n^{i-1})_{t_n^0 \ldots t_n^{i-1}})$. It is evident that $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ so defined satisfy the projection properties (3.4.2) and (3.4.3). Moreover it is easy to prove that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\{t_n^0, \ldots, t_n^i\}, \boldsymbol{\varphi})$: since $\pi^{(i)}(\boldsymbol{\delta}_{(z^0, \ldots, z^{i-1})}) = \boldsymbol{\delta}_{(z^0, \ldots, z^{i-1}, z^{i-1})}$ and $\tilde{\mathcal{T}}^2_{\nabla\varphi_n^i - \nabla\varphi_n^{i-1}}(\boldsymbol{\delta}_{(z, \nabla v)}) = \boldsymbol{\delta}_{(z, \nabla v + \nabla\varphi_n^i - \nabla\varphi_n^{i-1})}$, with $v + \varphi_n^i - \varphi_n^{i-1} \in \mathcal{A}(t_n^i)$ whenever $v \in \mathcal{A}(t_n^{i-1})$, applying Lemma 1.3.6 we can obtain the approximation properties (3.3.1) and (3.3.2). $\square$

LEMMA 3.4.2. *For every i the functional (3.4.1) has a minimizer over A_n^i.*

PROOF. Let $(\boldsymbol{\nu}^h, \boldsymbol{\mu}^h)_h \subset A_n^i$ be a minimizing sequence. By the bounds on W and the assumption on $\boldsymbol{l}$, using Poincaré inequality we have

$$\begin{aligned} c \int_{D \times \times \mathbb{R}^{\mathrm{N} \times \mathrm{d}}} [|\theta|^2 + |F|^2] \, d\boldsymbol{\nu}^h_{t_n^i}(x, \theta, F) - C'(1 + \|\nabla \boldsymbol{v}^h(t_n^i)\|_2) \leq \\ \leq \textstyle\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} W(\theta, F) \, d\boldsymbol{\nu}^h_{t_n^i}(x, \theta, F) - \langle l_n^i, \boldsymbol{v}^h(t_n^i) \rangle \leq C', \end{aligned}$$

for every h, for positive constants c, C'. Since by Remark 3.3.8

$$\|\nabla \boldsymbol{v}^h(t_n^i)\|_2 \leq \Big(\int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |F|^2 \, d\boldsymbol{\nu}^h_{t_n^i}(x, \theta, F) \Big)^{1/2}, \tag{3.4.5}$$

we can deduce that

$$\sup_h \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |(\theta, F)|^2 \, d\boldsymbol{\nu}^h_{t_n^i}(x, \theta, F) \leq C'. \tag{3.4.6}$$

Since, thanks to Remark 3.3.3

$$\begin{aligned} \int_{D \times (\mathbb{R}^m)^{i+1}} |(\theta_0, \ldots, \theta_i)|^2 \, d\boldsymbol{\mu}^h_{t_n^0 \ldots t_n^i}(x, \theta_0, \ldots, \theta_i) = \sum_{j=0}^{i+1} \int_{D \times \mathbb{R}^m} |\theta_j|^2 \, d\boldsymbol{\mu}^h_{t_n^j}(x, \theta_j) = \\ = \sum_{j=0}^{i+1} \int_{D \times \mathbb{R}^m \times \mathbb{R}^{N \times d}} |\theta_j|^2 \, d\boldsymbol{\nu}^h_{t_n^j}(x, F, \theta_j), \end{aligned}$$

the projection property (3.4.3) and (3.4.6) imply that the second moments of $\boldsymbol{\mu}^h_{t_n^0 \ldots t_n^i}$ are bounded uniformly with respect to h. From this and from (3.4.6) we can deduce that, up to a subsequence, there exist $\bar{\mu} \in Y^2(D;(\mathbb{R}^m)^{i+1})$ and $\bar{\nu} \in Y^2(D;\mathbb{R}^m \times \mathbb{R}^{N\times d})$ such that

$$\boldsymbol{\mu}^h_{t^0_n \dots t^i_n} \rightharpoonup \bar{\mu} \quad \text{2-weakly*}, \tag{3.4.7}$$

$$\boldsymbol{\nu}^h_{t^i_n} \rightharpoonup \bar{\nu} \quad \text{2-weakly*}. \tag{3.4.8}$$

Hence defining $\bar{\boldsymbol{\nu}}_{t^j_n} := (\boldsymbol{\nu}^{i-1}_n)_{t^j_n}$, for every $j < i$, $\bar{\boldsymbol{\nu}}_{t^i_n} := \bar{\nu}$, and $\bar{\boldsymbol{\mu}}$ as the unique element of $SY^2(\{t^0_n,\dots,t^i_n\},D;\mathbb{R}^m)$ such that $\bar{\boldsymbol{\mu}}_{t^0_n\dots t^i_n} = \bar{\mu}$, we obtain

$$(\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\mu}}) \in Y^2(D;\mathbb{R}^m \times \mathbb{R}^{N\times d})^{\{t^0_n,\dots,t^i_n\}} \times SY^2(\{t^0_n,\dots t^i_n\},D;\mathbb{R}^m).$$

Since the hypotheses of Lemma 3.3.6 are satisfied, we deduce that $(\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\mu}}) \in AY(\{t^0_n,\dots,t^i_n\},\boldsymbol{\varphi})$. Moreover, by construction, $(\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\mu}})$ satisfies also the projection properties (3.4.2) and (3.4.3), so we can conclude that $(\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\mu}}) \in A^i_n$.

By (1.3.3) the terms of (3.4.1) containing W and H are lower semicontinuous with respect to the 2-weak* convergence; on the other hand a variant of Lemma 1.3.11, with H^1_0 replaced by $\mathcal{A}(t^i_n)$, shows that the term of (3.4.1) containing l^i_n is continuous with respect to the 2-weak* convergence, therefore the functional (3.4.1) is 2-weakly* lower semicontinuous and this implies that $(\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\mu}})$ is a minimizer for it in A^i_n. □

Remark 3.4.3. Even if $W(\theta,\cdot)\colon \mathbb{R}^{N\times d} \to [0,+\infty)$ is convex for every $\theta \in \mathbb{R}^m$, it may happen that the incremental minimum problems have no solutions representable by functions.

We give an example in which this happens even for the first time step. By definition of A^1_n, there exists a solution of the first incremental minimum problem representable by function if and only if there exist $z_1 \in L^2(D;\mathbb{R}^m)$ and $v_1 \in \mathcal{A}(t^1_n)$ such that $((\boldsymbol{\delta}_{(z_0,\nabla v_0)},\boldsymbol{\delta}_{(z_1,\nabla v_1)}),\boldsymbol{\delta}_{(z_0,z_1)})$ realizes the minimum of the functional (3.4.1) on A^1_n. Consider the following case: $D = (0,1)^2$ and $N = m = 1$, $T = 1$, $\boldsymbol{l} \equiv 0$ and $\boldsymbol{\varphi}(t,(x_1,x_2)) := (1-t)x_1$, for every $t \in [0,1]$ and every $(x_1,x_2) \in (0,1)^2$. We consider

$$W(\theta,(F_1,F_2)) := |F_1 - a(\theta)|^2 + |F_2|^2 + b(\theta) + c, \tag{3.4.9}$$

where a is a $\mathcal{C}^1$ function satisfying $a(0) = a(1) = 1$ and $a(-1) = -1$, while b a $\mathcal{C}^1$ function such that $\tilde{b}(\theta) := b(\theta) + |\theta|$ is positive and vanishes only at 0, 1, and -1, and $c := -\inf b$. It can be easily verified that **(W.1)** and **(W.2)** are satisfied by suitable choices of a and b compatible with the requirements above. Now choose $H(\theta) := |\theta|$, $z_0 \equiv 0$ and $v_0(x_1,x_2) := x_1$. It is immediate to check that (z_0,v_0) satisfy the boundary conditions and (3.3.25). Moreover, by standard arguments, it can be easily shown that the infimum of functional in (3.4.1), for $i = 1$, is c and cannot be attained by functions which satisfy the boundary conditions. A minimizer of (3.4.1) on A^1_n is defined by

$$(\boldsymbol{\nu}^1_n)_{t^1_n} := \tfrac{t^1_n}{2}\boldsymbol{\delta}_{(-1,(-1,0))} + (1 - \tfrac{t^1_n}{2})\boldsymbol{\delta}_{(1,(1,0))},$$
$$(\boldsymbol{\mu}^1_n)_{t^0_n,t^1_n} := \tfrac{t^1_n}{2}\boldsymbol{\delta}_{(0,-1)} + (1 - \tfrac{t^1_n}{2})\boldsymbol{\delta}_{(0,1)}.$$

Set $\tau^n(s) := t^i_n$, whenever $t^i_n \le s < t^{i+1}_n$.

For every i and n we set

$$\sigma^i_n(x) := \int_{\mathbb{R}^m\times\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta,F)\; d(\boldsymbol{\nu}^i_n)^x_{t^i_n},$$

and define

$$\boldsymbol{\sigma}_n(t,x) := \sigma_n^i(x), \tag{3.4.10}$$

for a.e. $x \in D$, whenever $t_n^i \le t < t_n^{i+1}$.

We define $\boldsymbol{\nu}_n \in Y^2(D;\mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]}$ by

$$(\boldsymbol{\nu}_n)_s := (\boldsymbol{\nu}_n^i)_{t_n^i}, \tag{3.4.11}$$

whenever $t_n^i = \tau^n(s)$, for every $s \in [0,T]$; we define also $\boldsymbol{\mu}_n \in SY^2([0,T],D;\mathbb{R}^m)$ as the piecewise constant interpolation of $\boldsymbol{\mu}_n^{k(n)}$, as in Definition 2.1.8.

Note that $(\boldsymbol{\nu}_n,\boldsymbol{\mu}_n) \in AY([0,T],\boldsymbol{\varphi}(\tau^n(\cdot)))$ by construction.

3.4.2. A priori estimates. First of all we want to deduce a discrete version of the energy inequality for $(\boldsymbol{\nu}_n,\boldsymbol{\mu}_n)$.

Using the competitor defined in the proof of Lemma 3.4.1 and the fact that

$$\mathrm{bar}(\tilde\pi_{\mathbb{R}^{N\times d}}(\tilde{\mathcal{T}}^2_{\nabla\varphi_n^i-\nabla\varphi_n^{i-1}}((\boldsymbol{\nu}_n^i)_{t_n^i}))) = \mathrm{bar}(\tilde\pi_{\mathbb{R}^{N\times d}}((\boldsymbol{\nu}_n^i)_{t_n^i})) + \nabla\varphi_n^i - \nabla\varphi_n^{i-1},$$

for every i and n, we have

$$\begin{aligned}
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\theta,F) - \langle l_n^i, \boldsymbol{v}_n^i(t_n^i)\rangle + \\
&\quad + \int_{D\times(\mathbb{R}^m)^2} H(\theta_i-\theta_{i-1})\; d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x,\theta_{i-1},\theta_i) \le \\
&\le \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta, F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x))\; d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\theta,F) + \\
&\quad - \langle l_n^i, \boldsymbol{v}_n^i(t_n^{i-1}) + \varphi_n^i - \varphi_n^{i-1}\rangle + \\
&\quad + \int_{D\times(\mathbb{R}^m)^{i+1}} H(\theta_i-\theta_{i-1})\; d(\pi^{(i)}((\boldsymbol{\mu}_n^{i-1})_{t_n^0\dots t_n^{i-1}}))(x,\theta_0,\dots,\theta_{i-1},\theta_i).
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\theta,F) - \langle l_n^i, \boldsymbol{v}_n^i(t_n^i)\rangle + \\
&\quad + \int_{D\times(\mathbb{R}^m)^2} H(\theta_i-\theta_{i-1})\; d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x,\theta_{i-1},\theta_i) \le \\
&\le \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\theta,F) - \langle \boldsymbol{l}_n^{i-1}, \boldsymbol{v}_n^{i-1}(t_n^{i-1})\rangle + \\
&+ \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x)) - W(\theta,F)]\; d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\theta,F) + \\
&\quad - \langle l_n^i, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) + \varphi_n^i - \varphi_n^{i-1}\rangle + \langle \boldsymbol{l}_n^{i-1}, \boldsymbol{v}_n^{i-1}(t_n^{i-1})\rangle.
\end{aligned}$$

Let us fix $t \in [0,T]$ such that $t_n^j \le t < t_n^{j+1}$; using

$$\begin{aligned}
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x)) - W(\theta,F)]\; d(\boldsymbol{\nu}_n^{i-1})_{t_n^{i-1}}(x,\theta,F) = \\
&= \int_{t_n^{i-1}}^{t_n^i} \Big(\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta, F+\varepsilon^n(s,x))\nabla\dot{\boldsymbol{\varphi}}(s,x)\; d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\; ds,
\end{aligned}$$

where $\varepsilon^n(s,x) := \nabla\boldsymbol{\varphi}(s,x) - \nabla\boldsymbol{\varphi}(\tau^n(s),x)$, for every $s \in [0,T]$ and every $x \in D$, and

$$\langle l_n^i, \boldsymbol{v}_n^{i-1}(t_n^{i-1}) + \varphi_n^i - \varphi_n^{i-1}\rangle - \langle \boldsymbol{l}_n^{i-1}, \boldsymbol{v}_n^{i-1}(t_n^{i-1})\rangle =$$
$$= \int_{t_n^{i-1}}^{t_n^i} [\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle + \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_n^{i-1}(t_n^{i-1}) - \boldsymbol{\varphi}(\tau^n(s)) + \boldsymbol{\varphi}(s)\rangle]\, ds,$$

and iterating from 0 to j, we obtain

$$\begin{aligned}
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\, d(\boldsymbol{\nu}_n)_t(x,\theta,F) - \langle \boldsymbol{l}(\tau^n(t)), \boldsymbol{v}_n(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu}_n;0,t) \leq \\
&\leq \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0), v_0\rangle + \int_0^{\tau^n(t)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\, ds + \\
&- \int_0^{\tau^n(t)} [\langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_n(s)\rangle + \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle]\, ds + \qquad (3.4.12)\\
&+ \int_0^{\tau^n(t)} \Big(\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} \Big[\frac{\partial W}{\partial F}(\theta, F+\varepsilon^n(s,x)) - \frac{\partial W}{\partial F}(\theta,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(s)\, d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\, ds + \\
&+ \int_0^{\tau^n(t)} \langle \dot{\boldsymbol{l}}(s), \boldsymbol{\varphi}(\tau^n(s)) - \boldsymbol{\varphi}(s)\rangle\, ds.
\end{aligned}$$

From (3.4.12), we can deduce the following a priori estimates on $(\boldsymbol{\nu}_n, \boldsymbol{\mu}_n)$.

LEMMA 3.4.4. *There exists a positive constant C, such that*

$$\sup_n \sup_{t\in[0,T]} \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} |(\theta,F)|^2\, d(\boldsymbol{\nu}_n)_t(x,\theta,F) \leq C, \qquad (3.4.13)$$
$$\sup_n \mathrm{Var}_H(\boldsymbol{\mu}_n;0,T) \leq C. \qquad (3.4.14)$$

PROOF. Using the fact that $\sup_{t\in[0,T]} \|\boldsymbol{l}(t)\|_{(H^1)^*}$, $\int_0^T \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*}\, dt$, and $\int_0^T \|\dot{\boldsymbol{\varphi}}(t)\|_{H^1}\, dt$ are finite, the hypotheses on W and the inequality

$$\sup_{s\in[0,T]} \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} (|(\theta,F)|^2\, d(\boldsymbol{\nu}_n)_s(x,\theta,F) < \infty,$$

(since $\boldsymbol{\nu}_n$ are piecewise constant interpolations of Young measures with finite second moments), we can deduce from (3.4.12) that, for n sufficiently large,

$$\begin{aligned}
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} (|(\theta,F)|^2\, d(\boldsymbol{\nu}_n)_t(x,\theta,F) \leq \qquad (3.4.15)\\
&\leq \tilde{C} + \tilde{C} \sup_{s\in[0,T]} \Big(1 + \tilde{c}\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} |(\theta,F)|^2\, d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)^{1/2},
\end{aligned}$$

for suitable positive constants $\tilde{C}$ and $\tilde{c}$ independent of t and n (to estimate the terms in the third line in (3.4.12) we use (3.4.5), while the term in the fourth line of (3.4.12) can be treated using $\pi_D((\boldsymbol{\nu}_n)_s) = \mathcal{L}^d$ and Hölder inequality).

Since this can be repeated for every $t \in [0,T]$, we deduce (3.4.13). Inequality (3.4.14) comes now from (3.4.13) and (3.4.12). ☐

We can also deduce the following energy inequality for $(\boldsymbol{\nu}_n, \boldsymbol{\mu}_n)$.

LEMMA 3.4.5. *For every $t \in [0,T]$ we have*

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n)_t(x,\theta,F) - \langle \boldsymbol{l}(\tau^n(t)), \boldsymbol{v}_n(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu}_n;0,t) \leq$$
$$\leq \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0), v_0\rangle + \int_0^{\tau^n(t)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\; ds + \tag{3.4.16}$$
$$- \int_0^{\tau^n(t)} [\langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_n(s)\rangle + \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle]\; ds + \rho_n,$$

where $\rho_n \to 0$ as $n \to \infty$.

PROOF. Thanks to (3.4.12) it is enough to prove that

$$\rho_n^1 := \int_0^{\tau^n(t)} \Big(\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [\frac{\partial W}{\partial F}(\theta, F+\varepsilon^n(s,x)) - \frac{\partial W}{\partial F}(\theta,F)]\nabla\dot{\boldsymbol{\varphi}}(s,x)\, d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\; ds$$

and

$$\rho_n^2 := \int_0^{\tau^n(t)} \langle \dot{\boldsymbol{l}}(s), \boldsymbol{\varphi}(\tau^n(s)) - \boldsymbol{\varphi}(s)\rangle\; ds$$

tend to 0 as $n \to \infty$. Since $\boldsymbol{\varphi}$ is uniformly continuous on $[0,T]$ with values in $H^1(D;\mathbb{R}^N)$, it is immediate to see that $\rho_n^2 \to 0$ as $n \to \infty$. It remains to prove that, fixed $\delta > 0$, $\rho_n^1 < \delta$ for n sufficiently large.

We recall that, since $\nabla\dot{\boldsymbol{\varphi}} \in L^1([0,T];L^2(D;\mathbb{R}^{N\times d}))$, we can find a sequence $v_j \in \mathcal{C}([0,T];\mathcal{C}(\bar{D};\mathbb{R}^N))$ such that $\int_0^T \|v_j(t) - \nabla\dot{\boldsymbol{\varphi}}(t)\|_2\; dt \to 0$, as $j \to \infty$.

Since $\pi_D((\boldsymbol{\nu}_n)_s) = \mathcal{L}^d$, using **(W.2)**, we can deduce for every $M > 1$ and every $s \in [0,T]$

$$\int_{\{(x,\theta,F):|\theta|+|F|>M\}} \Big[\frac{\partial W}{\partial F}(\theta, F+\varepsilon^n(s,x)) - \frac{\partial W}{\partial F}(\theta,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(s)\; d(\boldsymbol{\nu}_n)_s(x,\theta,F) \leq$$
$$\leq C \int_{\{(x,\theta,F):|\theta|+|F|>M\}} 2(1+|\theta|+|F|)|\nabla\dot{\boldsymbol{\varphi}}(s,x) - v_j(s,x)|\; d(\boldsymbol{\nu}_n)_s(x,\theta,F) +$$
$$+C \int_{\{(x,\theta,F):|\theta|+|F|>M\}} 2(1+|\theta|+|F|)|v_j(s,x)|\; d(\boldsymbol{\nu}_n)_s(x,\theta,F) +$$
$$+C\|\varepsilon^n(s)\|_2\|\nabla\dot{\boldsymbol{\varphi}}(s)\|_2;$$

therefore, thanks to Lemma 3.4.4, for every j we have

$$\int_0^{\tau^n(t)}\Big(\int_{\{(x,\theta,F):|\theta|+|F|>M\}}\Big[\frac{\partial W}{\partial F}(\theta,F+\varepsilon^n(s,x))-\frac{\partial W}{\partial F}(\theta,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(s)\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\ ds\leq$$

$$\leq 8C\Big[\sup_{s\in[0,T]}\sup_n\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}}|(\theta,F)|^2\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big]^{1/2}\int_0^t\|\nabla\dot{\boldsymbol{\varphi}}(s)-v_j(s)\|_2\ ds+$$
$$+4T\sup_{s\in[0,T]}\sup_n\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}}|(\theta,F)|^2\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\frac{\|v_j\|_\infty}{M}+$$
$$+C\sup_{s\in[0,T]}\|\varepsilon^n(s)\|_2\int_0^T\|\nabla\dot{\boldsymbol{\varphi}}(s)\|_2\ ds.$$

Since $s\mapsto\boldsymbol{\varphi}(s)$ is continuous from $[0,T]$ into $H^1(D;\mathbb{R}^N)$, the term in the last line tends to 0 as $n\to\infty$. Therefore there exist $\bar{j}$ and $\bar{M}$ such that

$$\int_0^{\tau^n(t)}\Big(\int_{\{(x,\theta,F):|\theta|+|F|>\bar{M}\}}\Big[\frac{\partial W}{\partial F}(\theta,F+\varepsilon^n(s,x))-\frac{\partial W}{\partial F}(\theta,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(s)\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\ ds\leq$$
$$\leq\frac{\delta}{2},\tag{3.4.17}$$

for n sufficiently large.

We now consider the contribution of the integral on $\{(x,\theta,F):|\theta|+|F|\leq\bar{M}\}$.

For every $M>0$ and $r>0$, define

$$\omega_M(r):=\sup_{|(\theta,F)|\leq M,|(\theta',F')|\leq r}\Big|\frac{\partial W}{\partial F}(\theta+\theta',F+F')-\frac{\partial W}{\partial F}(\theta,F)\Big|.$$

Thanks to the continuity properties of $\frac{\partial W}{\partial F}$, for every M we have $\omega_M(r)\to 0$ as r tends to 0; moreover it is $\omega_M(r)\leq 2C(M+1)+Cr$.

In particular, for $\bar{M}$ chosen before, it is immediate that

$$\int_0^{\tau^n(t)}\Big(\int_{\{(x,\theta,F):|\theta|+|F|\leq\bar{M}\}}\Big[\frac{\partial W}{\partial F}(\theta,F+\varepsilon^n(s,x))-\frac{\partial W}{\partial F}(\theta,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(s)\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\ ds\leq$$
$$\int_0^t\Big(\int_D\omega_{\bar{M}}(|\varepsilon^n(s,x)|)|\nabla\dot{\boldsymbol{\varphi}}(s,x)|\ dx\Big)\ ds;$$

by the Dominated Convergence Theorem, we have

$$\|\omega_{\bar{M}}(|\varepsilon^n(s)|)\|_2\to 0,$$

for a.e. $s\in[0,T]$, as n tends to ∞; since we have the estimate

$$\|\omega_{\bar{M}}(|\varepsilon^n(s)|)\|_2\|\nabla\dot{\boldsymbol{\varphi}}(s)\|_2\leq C\|\nabla\dot{\boldsymbol{\varphi}}(s)\|_2(2\bar{M}+2+\|\varepsilon^n(s)\|_2),$$

for every $s \in [0,T]$ and $\sup_{s\in[0,T]} \|\varepsilon^n(s)\|_2 \to 0$ as $n \to \infty$, we can apply again the Dominated Convergence Theorem and obtain

$$\int_0^{\tau^n(t)} \Big(\int_{\{(x,\theta,F):|\theta|+|F|\le \bar{M}\}} \Big[\frac{\partial W}{\partial F}(\theta, F+\varepsilon^n(s,x)) - \frac{\partial W}{\partial F}(\theta,F)\Big] \nabla\dot{\boldsymbol{\varphi}}(s)\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\, ds \le \le \frac{\delta}{2}, \tag{3.4.18}$$

for n sufficiently large. Therefore (3.4.17) and (3.4.18) give the thesis. $\square$

3.4.3. Passage to the limit. Thanks to (3.4.13), (3.4.14), and hypothesis **(H.2)**, we can apply our version of Helly's Theorem (Theorem 2.1.6) to the sequence $\boldsymbol{\mu}_n$ and obtain a subsequence, still indicated by $\boldsymbol{\mu}_n$, a subset Θ of $[0,T]$, containing 0, with $\mathcal{L}^1([0,T]\backslash\Theta) = 0$, and $\boldsymbol{\mu} \in SY^2_-([0,T], D; \mathbb{R}^m)$, such that, for every finite sequence $t_1 < \cdots < t_l$ in Θ, we have

$$(\boldsymbol{\mu}_n)_{t_1\ldots t_l} \rightharpoonup \boldsymbol{\mu}_{t_1\ldots t_l}, \quad \text{2-weakly*}. \tag{3.4.19}$$

For every $t \in \Theta$ we choose an increasing sequence of integers n^t_k, possibly depending on t, such that

$$\limsup_n [\langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle + \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_n(t)\rangle] = \lim_k [\langle \boldsymbol{\sigma}_{n^t_k}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle + \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_{n^t_k}(t)\rangle], \tag{3.4.20}$$

where $\boldsymbol{v}_n(t)$ is defined as in Remark (3.3.8). Thanks to (3.4.13) and Lemma 2.1.9, there exists $\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]}$, such that $\tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$ for every $t \in [0,T]$, $\boldsymbol{\nu}$ is Θ-2-weakly*-approximable from the left, and satisfies the following property:

(conv) for every $t \in \Theta$, there exists a subsequence of $(\boldsymbol{\nu}_{n^t_k})_k$, still denoted by $(\boldsymbol{\nu}_{n^t_k})_k$, such that

$$(\boldsymbol{\nu}_{n^t_k})_t \rightharpoonup \boldsymbol{\nu}_t, \quad \text{2-weakly*}. \tag{3.4.21}$$

Note that the map (3.3.24) is measurable on $[0,T]$ since

$$\langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t)\rangle = \limsup_n [\langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_n(t)\rangle], \tag{3.4.22}$$

for every $t \in \Theta$, thanks to (3.4.20), **(W.2)**, Lemma 1.3.5, and Remark 3.3.8. Moreover, thanks to Lemma 3.4.4, we have

$$|\langle \boldsymbol{\sigma}_n(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_n(t)\rangle| \le C'[\|\nabla\dot{\boldsymbol{\varphi}}(t)\|_2 + \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*}];$$

therefore, thanks to the hypotheses on $\boldsymbol{l}$ and $\boldsymbol{\varphi}$, and to (3.4.22), the map (3.3.24) is integrable on $[0,T]$.

It can be shown that $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY([0,T], \boldsymbol{\varphi})$. Indeed, thanks to (3.4.19) and (3.4.21), we can apply Lemma 3.3.7 to get

$$(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\Theta, \boldsymbol{\varphi}). \tag{3.4.23}$$

Let now $t_1 < \cdots < t_l$ be a finite sequence in $[0,T]$. Since $\boldsymbol{\mu}$ is left continuous and $\boldsymbol{\nu}$ is Θ-2-weakly*-approximable from the left, for every $i=1,\dots,l$, there exist a sequence, $s_i^j \to t_i$ as $j\to\infty$, with $s_i^j \le t_i$ and $s_i^j \in \Theta$, such that

$$\boldsymbol{\mu}_{s_1^j\dots s_n^j} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_n}, \quad \text{2-weakly*}, \tag{3.4.24}$$

$$\boldsymbol{\nu}_{s_i^j} \rightharpoonup \boldsymbol{\nu}_{t_i}, \quad \text{2-weakly*}, \text{ for every } i=1,\dots,n \tag{3.4.25}$$

as $j\to\infty$. If we define $\boldsymbol{\varphi}^j\colon \{t_1,\dots,t_l\} \to H^1(D;\mathbb{R}^N)$ by $\boldsymbol{\varphi}^j(t_i) := \boldsymbol{\varphi}(s_i^j)$, for every $i=1,\dots,n$, we have that $\boldsymbol{\varphi}^j(t_i)\to\boldsymbol{\varphi}(t_i)$ strongly in $H^1(D;\mathbb{R}^N)$, for every $i=1,\dots,n$; if we define $(\tilde{\boldsymbol{\nu}}^j,\tilde{\boldsymbol{\mu}}^j) \in Y^2(D;\mathbb{R}^m\times\mathbb{R}^{N\times d})^{\{t_1,\dots,t_l\}} \times SY^2(\{t_1,\dots,t_l\},D;\mathbb{R}^m)$ by

$$\tilde{\boldsymbol{\nu}}^j_{t_i} := \boldsymbol{\nu}_{s_i^j}, \quad \text{for every } i=1,\dots,n,$$

$$\tilde{\boldsymbol{\mu}}^j_{t_1\dots t_l} := \boldsymbol{\mu}_{s_1^j\dots s_l^j},$$

thanks to (3.4.23), we have that $(\tilde{\boldsymbol{\nu}}^j,\tilde{\boldsymbol{\mu}}^j)\in AY(\{t_1,\dots,t_n\},\boldsymbol{\varphi}^j)$ and

$$(\tilde{\boldsymbol{\mu}}^j)_{t_1\dots t_l} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_l}, \quad \text{2-weakly*}$$

$$(\tilde{\boldsymbol{\nu}}^j)_{t_i} \rightharpoonup \boldsymbol{\nu}_{t_i}, \quad \text{2-weakly* for every } i=1,\dots,l,$$

as $j\to\infty$. Hence we are again in the hypotheses of Lemma 3.3.7 and we can conclude that $(\boldsymbol{\nu},\boldsymbol{\mu})\in AY([0,T],\boldsymbol{\varphi})$.

By construction, $(\boldsymbol{\nu},\boldsymbol{\mu})$ satisfies (ev0).

Now we want to prove that $(\boldsymbol{\nu},\boldsymbol{\mu})$ satisfies (ev1).

Let $\tilde z\in L^2(D;\mathbb{R}^m)$, $\tilde u\in H^1_{\Gamma_0}(0)$; for every n and for every $i=1,\dots,k(n)$, let consider the pair $(\hat{\boldsymbol{\nu}},\hat{\boldsymbol{\mu}})$, where $\hat{\boldsymbol{\nu}} := ((\boldsymbol{\nu}_n^i)_{t_n^0},\dots,(\boldsymbol{\nu}_n^i)_{t_n^{i-1}},\mathcal{T}_{(\tilde z,\nabla\tilde u)}((\boldsymbol{\nu}_n^i)_{t_n^i}))$ and $\hat{\boldsymbol{\mu}}$ is the unique compatible system in $SY^2(\{t_n^0,\dots,t_n^i\},D;\mathbb{R}^m)$ satisfying $\hat{\boldsymbol{\mu}}_{t_n^0\dots t_n^i} = \tilde{\mathcal{T}}^{i+1}_{\tilde z}((\boldsymbol{\mu}_n^i)_{t_n^0\dots t_n^i})$, with

$$\mathcal{T}_{(\tilde z,\nabla\tilde u)}(x,\theta,F) = (x,\theta+\tilde z(x),F+\nabla\tilde u(x))$$

and

$$\tilde{\mathcal{T}}^{i+1}_{\tilde z}(x,\theta_0,\dots,\theta_i) = (x,\theta_0,\dots,\theta_{i-1},\theta_i+\tilde z(x)).$$

This is an element of $AY(\{t_n^0,\dots,t_n^i\},\boldsymbol{\varphi})$ and satisfies (3.4.2) and (3.4.3), hence we can use it as a competitor to obtain

$$\begin{aligned}
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\theta,F) - \langle l_n^i, \boldsymbol{v}_n^i(t_n^i)\rangle + \\
&\qquad + \int_{D\times(\mathbb{R}^m)^{i+1}} H(\theta_i-\theta_{i-1})\; d(\boldsymbol{\mu}_n^i)_{t_n^0\dots t_n^i} \le \\
&\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta+\tilde z(x),F+\nabla\tilde u(x))\; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\theta,F) - \langle l_n^i, \boldsymbol{v}_n^i(t_n^i)+\tilde u\rangle + \\
&\qquad + \int_{D\times(\mathbb{R}^m)^{i+1}} H(\theta_i+\tilde z(x)-\theta_{i-1})\; d(\boldsymbol{\mu}_n^i)_{t_n^0\dots t_n^i}.
\end{aligned}$$

Thanks to the triangular inequality for H (which follows from **(H.1)**), this implies that

$$\begin{aligned}
\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F)-W(\theta+\tilde z(x),F+\nabla\tilde u(x))]\; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\theta,F) \le \\
\mathcal{H}(\tilde z) - \langle l_n^i,\tilde u\rangle.
\end{aligned}$$

By definition of $\boldsymbol{\nu}_n$, for every $t \in \Theta$ we obtain

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F) - W(\theta+\tilde{z}(x), F+\nabla\tilde{u}(x))]\ d(\boldsymbol{\nu}_n)_t(x,\theta,F) \leq \mathcal{H}(\tilde{z}) - \langle \boldsymbol{l}(\tau^n(t)), \tilde{u}\rangle$$

Since

$$|W(\theta,F) - W(\theta+\tilde{z}(x), F+\nabla\tilde{u}(x))| \leq \tilde{C}[(|\tilde{z}(x)| + |\nabla\tilde{u}(x)|)^2 + 1] + \tilde{C}(|\tilde{z}(x)| + |\nabla\tilde{u}(x)|)(|\theta| + |F|),$$

for a suitable positive constant $\tilde{C}$, and $(\boldsymbol{\nu}_{n_k^t})_t \rightharpoonup \boldsymbol{\nu}_t$ 2-weakly*, for a suitable subsequence $((\boldsymbol{\nu}_{n_k^t})_t)_k$, we can deduce from Lemma 1.3.5 that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F) - W(\theta+\tilde{z}(x), F+\nabla\tilde{u}(x))]\ d(\boldsymbol{\nu}_{n_k^t})_t(x,\theta,F) \to \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F) - W(\theta+\tilde{z}(x), F+\nabla\tilde{u}(x))]\ d\boldsymbol{\nu}_t(x,\theta,F),$$

as $n \to \infty$. Since $\boldsymbol{l}(\tau^n(t)) \to \boldsymbol{l}(t)$, strongly in $H^1(D;\mathbb{R}^N)^*$, for every $t \in [0,T]$, we obtain (ev1) for $t \in \Theta$.

Let now $t \in [0,T]\setminus\Theta$. We can find a sequence $s^j \to t$, with $s^j \leq t$ and $s^j \in \Theta$, such that $\boldsymbol{\nu}_{s^j} \rightharpoonup \boldsymbol{\nu}_t$ 2-weakly*; as before we have that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F) - W(\theta+\tilde{z}(x), F+\nabla\tilde{u}(x))]\ d\boldsymbol{\nu}_{s^j}(x,\theta,F) \to \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta,F) - W(\theta+\tilde{z}(x), F+\nabla\tilde{u}(x))]\ d\boldsymbol{\nu}_t(x,\theta,F),$$

as $j \to \infty$. Again, since $\boldsymbol{l}(s^j) \to \boldsymbol{l}(t)$ strongly in $H^1(D;\mathbb{R}^N)^*$, using (ev1) for s^j, we obtain (ev1) for every $t \in [0,T]$.

Now we want to prove (ev2). Fix $t \in \Theta$ and let $(\boldsymbol{\nu}_{n_k^t})_t$ be a subsequence satisfying (3.4.20) and such that $(\boldsymbol{\nu}_{n_k^t})_t \rightharpoonup \boldsymbol{\nu}_t$ 2-weakly* as $k \to \infty$; then, thanks to (3.4.19),

$$\boldsymbol{\mu}_{n_k^t} \rightharpoonup \boldsymbol{\mu}, \tag{3.4.26}$$

weakly* in $SY^2([0,T],D;\mathbb{R}^m)$; since the term containing $\boldsymbol{l}$ is continuous with respect to 2-weak* convergence, the term containing W is weakly lower semicontinuous and the variation is weakly lower semicontinuous thanks to (3.4.26) and Lemma 2.1.7, we have

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\ d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \\ +\mathrm{Var}_H(\boldsymbol{\mu};0,t) \leq \\ \leq \liminf_k \Big[\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\ d(\boldsymbol{\nu}_{n_t^k})_t(x,\theta,F) - \langle \boldsymbol{l}(\tau^{n_k^t}(t)), \boldsymbol{v}_{n_k^t}(t)\rangle + \\ +\mathrm{Var}_H(\boldsymbol{\mu}_{n_k^t};0,t)\Big].$$

Using (3.4.16) of Lemma 3.4.5, we can deduce that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t),\boldsymbol{v}(t)\rangle + \\ +\mathrm{Var}_H(\boldsymbol{\mu};0,t) \leq \\ \leq \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0),\boldsymbol{v}(0)\rangle - \int_0^t \langle \boldsymbol{l}(s),\dot{\boldsymbol{\varphi}}(s)\rangle\; ds + \\ + \liminf_k \int_0^{\tau^{n_k^t}(t)} [\langle \boldsymbol{\sigma}_{n_k^t}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{n_k^t}(s)\rangle]\; ds.$$

Since $\sup_{s\in[0,T]}\sup_n \|\boldsymbol{\sigma}_n(s)\|_2$ and $\sup_{s\in[0,T]}\sup_n \|\boldsymbol{v}_n(s)\|_2$ are finite, we can deduce, using Fatou Lemma, that

$$\liminf_k \int_0^{\tau^{n_k^t}(t)} [\langle \boldsymbol{\sigma}_{n_k^t}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{n_k^t}(s)\rangle]\; ds \leq \\ \leq \limsup_n \int_0^{\tau^n(t)} [\langle \boldsymbol{\sigma}_n(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_n(s)\rangle]\; ds \leq \\ \leq \int_0^t \limsup_n [\langle \boldsymbol{\sigma}_n(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_n(s)\rangle]\; ds.$$

Thanks to (3.4.22) this implies that

$$\liminf_k \int_0^{\tau^{n_k^t}(t)} [\langle \boldsymbol{\sigma}_{n_k^t}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_{n_k^t}(s)\rangle]\; ds \leq \\ \leq \int_0^t [\langle \boldsymbol{\sigma}(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}(s)\rangle]\; ds.$$

This implies (ev2) for $t\in\Theta$.

Let now $t\in[0,T]\setminus\Theta$ and let $s^j\to t$, $s_j\leq t$ and (2.1.2); it is easy to verify, using Lemma 2.1.7, that

$$\mathrm{Var}_H(\boldsymbol{\mu};0,t)\leq \liminf_j \mathrm{Var}_H(\boldsymbol{\mu};0,s^j),$$

hence (ev2) for t can be deduced from (ev2) for s^j.

3.5. An alternative proof of the existence result in the quasiconvex case

In this section we focus on the particular case of $W(\theta,\cdot)$ quasiconvex, which can be treated using a spatial regularization depending on the gradient of the internal variable, as proposed in [**21**].

We assume that $W(\theta,\cdot)$ is quasiconvex for every $\theta\in\mathbb{R}^m$.

Definition 3.5.1. Let $\eta>0$, $\boldsymbol{\varphi}\in AC([0,T];H^1(D;\mathbb{R}^N))$, $\boldsymbol{l}\in AC([0,T];H^1(D;\mathbb{R}^N)^*)$, $z_0\in H^1(D;\mathbb{R}^m)$, $v_0\in\mathcal{A}(0)$, and $T>0$. A *solution of the η-regularized problem* with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0,v_0), in the time interval $(0,T]$, is a pair $(\boldsymbol{z}_\eta,\boldsymbol{v}_\eta)$, with $\boldsymbol{z}_\eta\colon[0,T]\to H^1(D;\mathbb{R}^m)$ and $\boldsymbol{v}_\eta\colon[0,T]\to H^1(D;\mathbb{R}^N)$, satisfying the following properties:

(ev0)$_{\mathrm{reg}}$ *initial condition:* $(\boldsymbol{z}_\eta(0),\boldsymbol{v}_\eta(0))=(z_0,v_0)$;

$(\mathrm{ev1})_{\mathrm{reg}}$ *kinematic admissibility:* $\boldsymbol{v}_\eta(t) \in \mathcal{A}(t)$ for every $t \in [0,T]$;

$(\mathrm{ev2})_{\mathrm{reg}}$ *global stability*: for every $t \in (0,T]$ and $(\hat{z},\hat{v}) \in H^1(D;\mathbb{R}^m) \times \mathcal{A}(t)$ we have

$$\begin{aligned}&\mathcal{W}(\boldsymbol{z}_\eta(t),\boldsymbol{v}_\eta(t)) - \langle \boldsymbol{l}(t),\boldsymbol{v}_\eta(t)\rangle + \tfrac{\eta}{2}\|\nabla \boldsymbol{z}_\eta(t)\|_2^2 \leq \\ &\leq \mathcal{W}(\hat{z},\hat{v}) - \langle \boldsymbol{l}(t),\hat{v}\rangle + \tfrac{\eta}{2}\|\nabla \hat{z}\|_2^2 + \mathcal{H}(\hat{z}-\boldsymbol{z}_\eta(t));\end{aligned}$$

$(\mathrm{ev3})_{\mathrm{reg}}$ *energy inequality:* $\mathrm{Var}_H(\boldsymbol{z}_\eta;0,T) < \infty$, the map $t \mapsto [\langle \boldsymbol{\sigma}_\eta(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t),\boldsymbol{v}_\eta(t)\rangle]$ is measurable on $[0,T]$, and for every $t \in [0,T]$ we have

$$\begin{aligned}&\mathcal{W}(\boldsymbol{z}_\eta(t),\boldsymbol{v}_\eta(t)) - \langle \boldsymbol{l}(t),\boldsymbol{v}_\eta(t)\rangle + \tfrac{\eta}{2}\|\nabla \boldsymbol{z}_\eta(t)\|_2^2 + \mathrm{Var}_H(\boldsymbol{z}_\eta;0,t) \leq \\ &\leq \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0),v_0\rangle + \tfrac{\eta}{2}\|\nabla z_0\|_2^2 - \int_0^t \langle \boldsymbol{l}(s),\dot{\boldsymbol{\varphi}}(s)\rangle\, ds + \\ &+ \int_0^t [\langle \boldsymbol{\sigma}_\eta(s),\nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s),\boldsymbol{v}_\eta(s)\rangle]\, ds,\end{aligned}$$

where $\boldsymbol{\sigma}_\eta(s) := \frac{\partial W}{\partial F}(\boldsymbol{z}_\eta(s),\nabla\boldsymbol{v}_\eta(s))$ for every $s \in [0,T]$.

The proof of Theorem 4.1 in [**21**] can be repeated in our case to obtain the following result.

THEOREM 3.5.2. *Let η, $\boldsymbol{\varphi}$, $\boldsymbol{l}$, (z_0,v_0), and T be as in Definition 3.5.1. Then there exists a solution of the η-regularized problem with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0,v_0), in the time interval $(0,T]$. Moreover there exists a positive constant K_η such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_\eta(t)\|_2 \leq K_\eta, \tag{3.5.1}$$

$$\sup_{t\in[0,T]} \|\boldsymbol{v}_\eta(t)\|_{H^1} \leq K_\eta. \tag{3.5.2}$$

LEMMA 3.5.3. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, v_0, and T be as in Definition 3.5.1, and let $(z_0^\eta)_{\eta>0}$ be a family of functions in $H^1(D;\mathbb{R}^m)$, such that*

$$\sup_\eta \|z_0^\eta\|_2 \leq M, \tag{3.5.3}$$

$$\sup_\eta \eta\|\nabla z_0^\eta\|_2^2 \leq M, \tag{3.5.4}$$

for a suitable positive constant M. Then there exists a positive constant K such that the solutions $(\boldsymbol{z}_\eta,\boldsymbol{v}_\eta)$ of the η-regularized problems with initial condition (z_0^η,v_0) satisfy the following conditions

$$\sup_{t\in[0,T]}\sup_\eta \|\boldsymbol{z}_\eta(t)\|_2 \leq K, \qquad \sup_{t\in[0,T]}\sup_\eta \|\nabla\boldsymbol{v}_\eta(t)\|_2 \leq K, \tag{3.5.5}$$

$$\sup_\eta \mathrm{Var}_H(\boldsymbol{z}_\eta;0,T) \leq K, \qquad \sup_{t\in[0,T]} \tfrac{\eta}{2}\|\nabla\boldsymbol{z}_\eta(t)\|_2^2 \leq K. \tag{3.5.6}$$

PROOF. Using the fact that $\sup_{t\in[0,T]}\|\boldsymbol{l}(t)\|_{(H^1)^*}$, $\int_0^T \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*}\, dt$, and $\int_0^T \|\dot{\boldsymbol{\varphi}}(t)\|_{H^1}\, dt$ are finite, the hypotheses on W, the hypotheses (3.5.3) and (3.5.4), and the inequalities (3.5.1) and (3.5.2), we can deduce from $(\mathrm{ev3})_{\mathrm{reg}}$ that, for every $\eta > 0$,

$$c(\|\boldsymbol{z}_\eta(t)\|_2^2 + \|\nabla\boldsymbol{v}_\eta(t)\|_2^2) - C \leq \tilde{C} + \tilde{c}\sup_{s\in[0,T]}(\|\boldsymbol{z}_\eta(s)\|_2 + \|\boldsymbol{v}_\eta(s)\|_{H^1}),$$

for suitable positive constants $\tilde{c}, \tilde{C}$ independent of t and η. Since this can be repeated for every $t \in [0,T]$, Poincaré inequality implies (3.5.5). Inequalities (3.5.6) come now from $(\mathrm{ev3})_{\mathrm{reg}}$ and (3.5.5). $\square$

REMARK 3.5.4. From (3.5.6) we can deduce that

$$\eta_n \nabla \boldsymbol{z}_{\eta_n}(t) \to 0$$

strongly in $L^2(D;\mathbb{R}^{m\times d})$, for every positive sequence $\eta_n \to 0$ and every $t \in [0,T]$.

DEFINITION 3.5.5. Given an external load $\boldsymbol{l} \in AC([0,T];H^1(D;\mathbb{R}^N)^*)$, a boundary datum $\boldsymbol{\varphi} \in AC([0,T];H^1(D;\mathbb{R}^N))$, an initial condition $(z_0, v_0) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(0)$, and $T > 0$, a *quasistatic evolution obtained by spatial regularizations* in the time interval $[0,T]$ is a pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D;\mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D;\mathbb{R}^m)$ with

$$\tilde{\pi}_{\mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t, \quad \text{for every } t \in [0,T], \tag{3.5.7}$$

for which there exist:

- a sequence $(z_0^n)_n \subset H^1(D;\mathbb{R}^m)$, with
$$z_0^n \to z_0 \quad \text{strongly in } L^2(D;\mathbb{R}^m), \tag{3.5.8}$$
- a positive sequence η_n converging to 0, with
$$\eta_n \|\nabla z_0^n\|_2^2 \to 0, \tag{3.5.9}$$
- a subset Θ of $[0,T]$, with $0 \in \Theta$ and $\mathcal{L}^1([0,T] \setminus \Theta) = 0$,

such that $\boldsymbol{\nu}$ is Θ-2-weakly* approximable from the left and the solutions $(\boldsymbol{z}_{\eta_n}, \boldsymbol{v}_{\eta_n})$ of the η_n-regularized problems with initial conditions (z_0^n, v_0) in the time interval $(0,T]$ satisfy the following conditions:

$(\mathrm{conv1})_{\mathrm{reg}}$ for every finite sequence $t_1 < \cdots < t_l$ in Θ
$$\boldsymbol{\delta}_{(\boldsymbol{z}_{\eta_n}(t_1),\ldots,\boldsymbol{z}_{\eta_n}(t_l))} \rightharpoonup \boldsymbol{\mu}_{t_1\ldots t_l} \quad \text{2-weakly*}, \tag{3.5.10}$$
as $n \to \infty$;

$(\mathrm{conv2})_{\mathrm{reg}}$ for every $t \in \Theta$ there exists a subsequence $(\boldsymbol{z}_{\eta_{n_k^t}}, \boldsymbol{v}_{\eta_{n_k^t}})_k$ of $(\boldsymbol{z}_{\eta_n}, \boldsymbol{v}_{\eta_n})_n$, possibly depending on t, with
$$\begin{aligned}\limsup_n\, &[\langle \boldsymbol{\sigma}_{\eta_n}(t), \nabla\boldsymbol{\varphi}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_{\eta_n}(t)\rangle] = \\ &= \lim_k\, [\langle \boldsymbol{\sigma}_{\eta_{n_k^t}}(t), \nabla\boldsymbol{\varphi}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_{\eta_{n_k^t}}(t)\rangle],\end{aligned} \tag{3.5.11}$$
and
$$\boldsymbol{\delta}_{(\boldsymbol{z}_{\eta_{n_k^t}}(t), \nabla \boldsymbol{v}_{\eta_{n_k^t}}(t))} \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly*}, \tag{3.5.12}$$
as $k \to \infty$.

In the next theorem we will show that the quasistatic evolution obtained by spatial regularizations fulfils the requirements of Definition 3.3.14.

THEOREM 3.5.6. *Let* $\boldsymbol{l} \in AC([0,T];H^1(D;\mathbb{R}^N)^*)$, $\boldsymbol{\varphi} \in AC([0,T];H^1(D;\mathbb{R}^N))$, (z_0, v_0), *and* $T > 0$ *be as in Definition 3.5.5, and assume that*

$$\mathcal{W}(z_0, v_0) \le \mathcal{W}(\hat{z}, \hat{v}) - \langle \boldsymbol{l}(0), \hat{v} - v_0\rangle + \mathcal{H}(\hat{z} - z_0), \tag{3.5.13}$$

for every $\hat{z} \in L^2(D;\mathbb{R}^m)$ *and every* $\hat{v} \in \mathcal{A}(0)$. *Then a quasistatic evolution obtained by spatial regularizations in the time interval* $[0,T]$ *is a globally stable quasistatic evolution of Young measures with the same data in the time interval* $[0,T]$.

PROOF. Let $(\boldsymbol{\nu},\boldsymbol{\mu}) \in Y^2(D;\mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T],D;\mathbb{R}^m)$ be a quasistatic evolution obtained by spatial regularizations.

First of all we show that $(\boldsymbol{\nu},\boldsymbol{\mu}) \in AY([0,T],\boldsymbol{\varphi})$: thanks to (3.5.10) and (3.5.12), it is immediate to see that $(\boldsymbol{\nu},\boldsymbol{\mu}) \in AY(\Theta,\boldsymbol{\varphi})$; as in the proof of the main theorem, since $\boldsymbol{\nu}$ is Θ-2-weakly*-approximable from the left and $\boldsymbol{\mu}$ is left continuous, we are able to prove that $(\boldsymbol{\nu},\boldsymbol{\mu}) \in AY([0,T],\boldsymbol{\varphi})$. In particular, for every $t \in [0,T]$ there exists a function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that $\nabla \boldsymbol{v}(t) = \mathrm{bar}(\tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$, thanks to Remark 3.3.8.

Let η_n and z_0^n be the sequences appearing in Definition 3.5.5, and let $(\boldsymbol{z}_{\eta_n},\boldsymbol{v}_{\eta_n})$ be the solutions of the η_n-regularized problems in the time interval $(0,T]$; (ev0) comes immediately from $(\mathrm{ev0})_{\mathrm{reg}}$, (3.5.8), and (3.5.12).

Now we prove that $(\boldsymbol{\nu},\boldsymbol{\mu})$ satisfies (ev1).

For $t=0$ it comes immediately from (3.5.13).

For $t \neq 0$, we first show that (ev1) holds for test functions $\tilde{z} \in H^1(D;\mathbb{R}^m)$. Let $t \in \Theta\setminus 0$, $(\eta_{n_k^t})_k$ be the sequence appearing in $(\mathrm{conv2})_{\mathrm{reg}}$, and $(\tilde{z},\tilde{u}) \in H^1(D;\mathbb{R}^m) \times H^1_{\Gamma_0}(0)$. By $(\mathrm{ev1})_{\mathrm{reg}}$, we have

$$\mathcal{W}(\boldsymbol{z}_{\eta_{n_k^t}}(t),\boldsymbol{v}_{\eta_{n_k^t}}(t)) + \tfrac{\eta_{n_k^t}}{2}\|\nabla \boldsymbol{z}_{\eta_{n_k^t}}(t)\|_2^2 \leq$$
$$\leq \mathcal{W}(\boldsymbol{z}_{\eta_{n_k^t}}(t)+\tilde{z},\boldsymbol{v}_{\eta_{n_k^t}}(t)+\tilde{u}) - \langle \boldsymbol{l}(t),\tilde{u}\rangle + \frac{\eta_{n_k^t}}{2}\|\nabla \boldsymbol{z}_{\eta_{n_k^t}}(t)+\nabla\tilde{z}\|_2^2 + \mathcal{H}(\tilde{z}).$$

Thanks to Remark 3.5.4,

$$\frac{\eta_{n_k^t}}{2}\|\nabla \boldsymbol{z}_{\eta_{n_k^t}}(t)+\nabla\tilde{z}\|_2^2 - \tfrac{\eta_{n_k^t}}{2}\|\nabla \boldsymbol{z}_{\eta_{n_k^t}}(t)\|_2^2 = \frac{\eta_{n_k^t}}{2}\|\nabla\tilde{z}\|_2^2 + \langle \eta_{n_k^t}\nabla \boldsymbol{z}_{\eta_{n_k^t}}(t),\nabla\tilde{z}\rangle, \to 0,$$

as $k\to\infty$. On the other hand we have

$$\mathcal{W}(\boldsymbol{z}_{\eta_{n_k^t}}(t)+\tilde{z},\boldsymbol{v}_{\eta_{n_k^t}}(t)+\tilde{u}) - \mathcal{W}(\boldsymbol{z}_{\eta_{n_k^t}}(t),\boldsymbol{v}_{\eta_{n_k^t}}(t)) =$$
$$= \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta+\tilde{z}(x),F+\nabla\tilde{u}(x)) - W(\theta,F)]\, d\boldsymbol{\delta}_{(\boldsymbol{z}_{\eta_{n_k^t}}(t),\nabla \boldsymbol{v}_{\eta_{n_k^t}}(t))}(x,\theta,F);$$

since

$$|W(\theta+\tilde{z}(x),F+\nabla\tilde{u}(x)) - W(\theta,F)| \leq \tag{3.5.14}$$
$$\tilde{C}[(|\tilde{z}(x)|+|\nabla\tilde{u}(x)|)^2+1] + \tilde{C}(|\tilde{z}(x)|+|\nabla\tilde{u}(x)|)(|\theta|+|F|),$$

thanks to Lemma 1.3.5 and (3.5.12) we can deduce that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta+\tilde{z}(x),F+\nabla\tilde{u}(x)) - W(\theta,F)]\, d\boldsymbol{\delta}_{(\boldsymbol{z}_{\eta_{n_k^t}}(t),\nabla \boldsymbol{v}_{\eta_{n_k^t}}(t))}(x,\theta,F) \to$$
$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta+\tilde{z}(x),F+\nabla\tilde{u}(x)) - W(\theta,F)]\, d\boldsymbol{\nu}_t(x,\theta,F),$$

and hence we can conclude that

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\; d\boldsymbol{\nu}_t(x,\theta,F) \leq \tag{3.5.15}$$
$$\leq \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta+\tilde{z}(x),F+\nabla\tilde{u}(x))\; d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t),\tilde{u}\rangle + \mathcal{H}(\tilde{z}).$$

Using the fact that $\boldsymbol{\nu}$ is Θ-2-weakly*-approximable from the left, it is easy to extend the previous inequality to $t\in[0,T]\setminus\Theta$ reasoning as in the proof of the main theorem.

Now we prove that (3.5.15) holds also for test functions $\tilde{z}\in L^2(D;\mathbb{R}^m)$. Let $t\in(0,T]$, $\tilde{z}\in L^2(D;\mathbb{R}^m)$, $\tilde{u}\in H^1_{\Gamma_0}(0)$, and assume that $(\tilde{z}_h)_h$ is a sequence in $H^1(D;\mathbb{R}^m)$ with $\tilde{z}_h\to\tilde{z}$ strongly in $L^2(D;\mathbb{R}^m)$; then we have

$$\mathcal{H}(\tilde{z}_h)\to\mathcal{H}(\tilde{z}),$$

and

$$\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} [W(\theta+\tilde{z}(x),F+\nabla\tilde{u}(x)) - W(\theta+\tilde{z}_h(x),F+\nabla\tilde{u}(x))]\; d\boldsymbol{\nu}_t(x,\theta,F)\to 0,$$

thanks to the hypotheses on H, and to **(W.2)** and (1.3.2), respectively; hence (ev1) for test functions $\tilde{z}$ and $\tilde{u}$ can be deduce from (3.5.15) applied to $\tilde{z}_h$ and $\tilde{u}$.

Finally we prove (ev2).

Thanks to (3.5.8) and (3.5.9), we have

$$\mathcal{W}(z_0^n,v_0)+\tfrac{\eta_n}{2}\|\nabla z_0^n\|_2^2\to\mathcal{W}(z_0,v_0),$$

as $n\to\infty$; therefore we can argue as in the proof of the main theorem using $(\mathrm{ev2})_{\mathrm{reg}}$ and the approximation properties of $(\boldsymbol{\nu},\boldsymbol{\mu})$. □

THEOREM 3.5.7. *Let* $\boldsymbol{l}\in AC([0,T];H^1(D;\mathbb{R}^N)^*)$, $\boldsymbol{\varphi}\in AC([0,T];H^1(D;\mathbb{R}^N))$, $T>0$, *and* $(z_0,v_0)\in L^2(D;\mathbb{R}^m)\times\mathcal{A}(0)$. *Then there exists a quasistatic evolution obtained by spatial regularizations with external load* $\boldsymbol{l}$, *boundary datum* $\boldsymbol{\varphi}$, *and initial condition* (z_0,v_0), *in the time interval* $[0,T]$.

PROOF. Fix a sequence $(z_0^n)_n$ in $H^1(D;\mathbb{R}^m)$ satisfying (3.5.8) and a positive sequence $\eta_n\to 0$ satisfying (3.5.9); let $(\boldsymbol{z}_{\eta_n},\boldsymbol{v}_{\eta_n})$ be the solutions of the η_n-regularized problems with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0^n,v_0). Thanks to (3.5.5), (3.5.6), and **(H.2)**, we can apply Theorem 2.1.6 to obtain a subset Θ of $[0,T]$ with $0\in\Theta$ and $\mathcal{L}^1([0,T]\setminus\Theta)=0$, a compatible system $\boldsymbol{\mu}\in SY^2_-([0,T],D;\mathbb{R}^m)$, and a subsequence still indicated by $(\eta_n)_n$ satisfying $(\mathrm{conv1})_{\mathrm{reg}}$. For every $t\in\Theta$ we select a subsequence $(\eta_{n_k^t})_k$ of $(\eta_n)_n$, possibly depending on t, satisfying (3.5.11). Thanks to (3.5.5), we can apply Lemma 2.1.9 to obtain another subsequence still denoted by $(\eta_{n_k^t})_k$ and $\boldsymbol{\nu}\in Y^2(D;\mathbb{R}^m\times\mathbb{R}^{N\times d})^{[0,T]}$ which is Θ-2-weakly*-approximable from the left and fulfils conditions $(\mathrm{conv2})_{\mathrm{reg}}$ and (3.5.7). □

CHAPTER 4

A vanishing viscosity approach

4.1. Introduction

In the previous chapter we have highlighted that the nonconvexity of the energy functional is responsible for the ill-posedness of the incremental minimum problems used in the construction of approximate solutions to the quasistatic evolution problem.

Moreover, since the lack of convexity allows the functional to have multiple wells, a quasistatic evolution driven by global minimizers, if they would exist, could prescribe abrupt jumps from one well to another one; therefore it is preferable to follow a path composed by local minimizers rather than global minimizers.

To this end, in the present chapter we propose a notion of quasistatic evolution in which the stability condition is weakened, it is indeed substituted by a stationarity condition. Moreover, we propose a selection criterion based on a sort of viscous approximation, among the evolutions satisfying the stationarity condition and the energy inequality. The underlying idea is that an evolution obtained in this way does not jump over potential barriers.

We consider viscously-regularized problems and prove an existence and uniqueness result for their solutions in $H^1([0,T];L^2(D:\mathbb{R}^m)\times H^1(D;\mathbb{R}^N))$.

Then, we send the regularizing parameter to zero and analyze the limit of the solutions to the regularized problems, in the framework of Young measures: due to the nonconvexity of the problem, the regularized solutions may develop stronger and stronger oscillations in space as the parameter goes to zero, and this prevents the passage to the limit of the stationarity condition and the energy inequality in the usual function spaces. Therefore, a weaker formulation of these conditions in terms of Young measures, or equivalently in terms of stochastic processes, is provided; we prove that the limit of the regularized solutions satisfies these properties, and hence it can be considered as a solution of our quasistatic evolution problem.

Finally, we compare the notion of approximable quasistatic evolution considered in the present chapter with the quasistatic evolution based on global minimization defined in the previous one. More precisely we study an example in which the approximable quasistatic evolution is unique and is a classical function which can be described explicitely. Theorem 4.7.2 proves that this evolution cannot be a globally stable quasistatic evolution, since it does not satisfy the stability condition.

4.2. The mechanical model

The reference configuration D satisfies the same assumptions as in Section 3.2. As before, v denotes the deformation, z is the internal variable, $W:\mathbb{R}^m\times\mathbb{R}^{N\times d}\to[0,+\infty)$

is the stored energy density, $H\colon \mathbb{R}^m \to [0,+\infty)$ the dissipation rate density, and $\mathcal{W}$ and $\mathcal{H}$ represent the integral functionals associated to W and H, respectively.

We assume that H satisfies the same assumptions **(H.1)** and **(H.2)** as in Section 3.2, while W is required to fulfil condition **(W.1)** of Section 3.2 and the following condition, which is quite stronger than **(W.2)**:

(W.3) W is of class $\mathcal{C}^2$ and there exists a positive constant M such that

$$\Big|\frac{\partial^2 W}{\partial(\theta,F)^2}(\theta,F)\Big| \le M,$$

for every $\theta \in \mathbb{R}^m$ and every $F \in \mathbb{R}^{N\times d}$, where $\frac{\partial^2 W}{\partial(\theta,F)^2}$ denotes the matrix of all second derivatives with respect to (the components of) θ and F.

The global dissipation $\mathrm{Var}_H(\boldsymbol{z};s,t)$ of a possibly discontinuous function $\boldsymbol{z}\colon [0,T] \to L^2(D;\mathbb{R}^m)$ in the interval $[s,t]$ is defined as in Section 3.2. Note that for $\boldsymbol{z} \in H^1([0,T];L^2(D;\mathbb{R}^m))$ we have

$$\mathrm{Var}_H(\boldsymbol{z};s,t) = \int_s^t \mathcal{H}(\dot{\boldsymbol{z}}(\tau))\,d\tau \tag{4.2.1}$$

(see, e.g., [**9**, Theorem 7.1]).

The external load at time t and the prescribed boundary datum on Γ_0 at time t are denoted by $\boldsymbol{l}(t)$ and $\boldsymbol{\varphi}(t)$, respectively; we assume that $\boldsymbol{l} \in H^1([0,T];H^1(D;\mathbb{R}^N)^*)$, where $H^1(D;\mathbb{R}^N)^*$ is the dual of $H^1(D;\mathbb{R}^N)$, and $\boldsymbol{\varphi} \in H^1([0,T];H^1(D;\mathbb{R}^N))$. As before, the kinematically admissible values at time t are contained in $L^2(D;\mathbb{R}^m)\times\mathcal{A}(t)$, where

$$\mathcal{A}(t) = H^1_{\Gamma_0}(\boldsymbol{\varphi}(t)) := \{v \in H^1(D;\mathbb{R}^N) \text{ such that } v = \boldsymbol{\varphi}(t) \ \ \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_0\}.$$

4.3. Regularized evolution

In this section we give the definition and an existence result for the solution of the ε-regularized evolution problem.

We will assume that the initial condition $(z_0,v_0) \in L^2(D;\mathbb{R}^m)\times\mathcal{A}(0)$ satisfies the following condition

$$\langle\sigma_0,\nabla\tilde u\rangle = \langle\boldsymbol{l}(0),\tilde u\rangle \quad \text{for every } \tilde u \in H^1_{\Gamma_0}(0), \tag{4.3.1}$$

$$\zeta_0 \in \partial\mathcal{H}(0), \tag{4.3.2}$$

where $\sigma_0(x) := \frac{\partial W}{\partial F}(z_0(x),\nabla v_0(x))$, $\zeta_0(x) := -\frac{\partial W}{\partial\theta}(z_0(x),\nabla v_0(x))$, for a.e. $x \in D$.

Definition 4.3.1. Let $\varepsilon > 0$, $\boldsymbol{l} \in H^1([0,T];H^1(D;\mathbb{R}^N)^*)$, $\boldsymbol{\varphi} \in H^1([0,T];H^1(D;\mathbb{R}^N))$, $(z_0,v_0) \in L^2(D;\mathbb{R}^m)\times\mathcal{A}(0)$, and $T>0$. Assume that (z_0,v_0) satisfies (4.3.1) and (4.3.2). A *solution of the ε-regularized problem* in the time interval $[0,T]$, with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0,v_0) is a pair $(\boldsymbol{z}_\varepsilon,\boldsymbol{v}_\varepsilon) \in H^1([0,T];L^2(D;\mathbb{R}^m)\times H^1(D;\mathbb{R}^N))$, satisfying the following conditions:

(ev0)$_\varepsilon$ *initial condition:* $(\boldsymbol{z}_\varepsilon(0),\boldsymbol{v}_\varepsilon(0)) = (z_0,v_0)$;

(ev1)$_\varepsilon$ *kinematic admissibility:* $\boldsymbol{v}_\varepsilon(t) \in \mathcal{A}(t)$, for every $t\in[0,T]$;

(ev2)$_\varepsilon$ *equilibrium condition:* for a.e. $t \in [0,T]$ and for every $\tilde u \in H^1_{\Gamma_0}(0)$,

$$\langle\boldsymbol{\sigma}_\varepsilon(t) + \varepsilon\nabla\dot{\boldsymbol{v}}_\varepsilon(t),\nabla\tilde u\rangle = \langle\boldsymbol{l}(t),\tilde u\rangle, \tag{4.3.3}$$

where $\boldsymbol{\sigma}_\varepsilon(t) := \frac{\partial W}{\partial F}(\boldsymbol{z}_\varepsilon(t),\nabla\boldsymbol{v}_\varepsilon(t))$;

$(\mathrm{ev}\hat{3})_\varepsilon$ *regularized flow rule:* for a.e. $t \in [0,T]$,

$$\dot{\boldsymbol{z}}_\varepsilon(t) = N_K^\varepsilon(\boldsymbol{\zeta}_\varepsilon(t)) \quad \text{a.e. in } D, \tag{4.3.4}$$

where $\boldsymbol{\zeta}_\varepsilon(t) := -\frac{\partial W}{\partial \theta}(\boldsymbol{z}_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t))$.

REMARK 4.3.2. If $\boldsymbol{l}$ can be written in the form

$$\langle \boldsymbol{l}(t), \tilde{v}\rangle = \int_D f(t,x)\cdot \tilde{v}(x)\; dx + \int_{\partial D} g(t,x)\cdot \tilde{v}(x)\; d\mathcal{H}^{d-1}(x),$$

with $f(t) \in L^2(D;\mathbb{R}^N)$ and $g(t) \in L^2(\partial D;\mathbb{R}^N)$, for every $\tilde{v} \in H^1(D;\mathbb{R}^N)$, then $(\mathrm{ev2})_\varepsilon$ takes the form

$$-\mathrm{div}\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) = f(t), \tag{4.3.5}$$

$$[\boldsymbol{\sigma}_\varepsilon(t) + \varepsilon\nabla\dot{\boldsymbol{v}}_\varepsilon(t)]\cdot \nu = g(t) \quad \mathcal{H}^{d-1}\text{-a.e. on } \Gamma_1, \tag{4.3.6}$$

where ν is the outer unit normal to ∂D. Indeed, choosing first $\tilde{u} \in H_0^1(D;\mathbb{R}^N)$ in $(\mathrm{ev2})_\varepsilon$, we obtain (4.3.5) and this ensures that $-\mathrm{div}\boldsymbol{\sigma}_\varepsilon(t) - \varepsilon\Delta\dot{\boldsymbol{v}}_\varepsilon(t) \in L^2(D;\mathbb{R}^N)$; hence we can apply integration by parts to get (4.3.6).

REMARK 4.3.3. Let us fix $t \in [0,T]$ such that $\dot{\boldsymbol{z}}_\varepsilon(t)$ and $\dot{\boldsymbol{v}}_\varepsilon(t)$ exist. Then the following conditions are equivalent

$$\dot{\boldsymbol{z}}_\varepsilon(t) = N_K^\varepsilon(\boldsymbol{\zeta}_\varepsilon(t)) \quad \text{a.e. in} \quad D, \tag{4.3.7}$$

$$\boldsymbol{\zeta}_\varepsilon(t) \in \partial\mathcal{H}_\varepsilon(\dot{\boldsymbol{z}}_\varepsilon(t)), \tag{4.3.8}$$

$$\boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t) \in \partial\mathcal{H}(\dot{\boldsymbol{z}}_\varepsilon(t)). \tag{4.3.9}$$

Indeed, by (1.1.5), $\partial\mathcal{H}_\varepsilon^*(\boldsymbol{\zeta}_\varepsilon(t)) = N_K^\varepsilon(\boldsymbol{\zeta}_\varepsilon(t))$, so that (4.3.7) and (4.3.8) are equivalent by standard property of conjugate functions (see, e.g., [**16**, Corollary I.5.2]). The equivalence of (4.3.8) and (4.3.9) comes from the definition of $\mathcal{H}_\varepsilon$.

In the following Theorem we prove that the modified flow rule $(\mathrm{ev}\hat{3})_\varepsilon$ can be replaced by a suitable constraint on $\boldsymbol{\zeta}_\varepsilon$ and an energy equality.

THEOREM 4.3.4. *Let $\boldsymbol{l}$, $\boldsymbol{\varphi}$, z_0, v_0, ε, and T be as in Definition (4.3.1).*

Then $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon) \in H^1([0,T]; L^2(D;\mathbb{R}^m)\times H^1(D;\mathbb{R}^N))$ is a solution of the ε-regularized problem in the time interval $[0,T]$, with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0,v_0) if and only if it satisfies the initial condition $(\mathrm{ev0})_\varepsilon$, the kinematic admissibility $(\mathrm{ev1})_\varepsilon$, the equilibrium condition $(\mathrm{ev2})_\varepsilon$, and the following conditions:

$(\mathrm{ev3})_\varepsilon$ *relaxed dual constraint: $\boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t) \in \partial\mathcal{H}(0)$, for a.e. $t\in[0,T]$;*

$(\mathrm{ev4})_\varepsilon$ *energy equality: for every $t \in [0,T]$,*

$$\mathcal{W}(\boldsymbol{z}_\varepsilon(t),\boldsymbol{v}_\varepsilon(t)) - \langle \boldsymbol{l}(t), \boldsymbol{v}_\varepsilon(t)\rangle + \int_0^t \mathcal{H}(\dot{\boldsymbol{z}}_\varepsilon(s))\,ds + \varepsilon\int_0^t \|\dot{\boldsymbol{z}}_\varepsilon(s)\|_2^2\,ds +$$
$$+\,\varepsilon\int_0^t \langle \nabla\dot{\boldsymbol{v}}_\varepsilon(s), \nabla\dot{\boldsymbol{v}}_\varepsilon(s) - \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds = \mathcal{W}(z_0,v_0) - \langle \boldsymbol{l}(0), v_0\rangle +$$
$$+\int_0^t \langle \boldsymbol{\sigma}_\varepsilon(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\,ds - \int_0^t [\langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_\varepsilon(s)\rangle + \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle]\,ds.$$

REMARK 4.3.5. Since we are dealing with the ε-regularized problem, in the *energy equality* there are two extra terms proportional to ε and depending on the time derivatives of $\boldsymbol{z}_\varepsilon$ and $\nabla \boldsymbol{v}_\varepsilon$; they represent a sort of viscous dissipation.

PROOF OF THEOREM 4.3.4. Suppose that $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ satisfies (ev0)$_\varepsilon$, (ev1)$_\varepsilon$, (ev2)$_\varepsilon$ and $(\mathrm{ev}\hat{3})_\varepsilon$. As $\mathcal{H}$ is positively homogeneous of degree one, by a general property of integral functionals (see, e.g., [**16**, Proposition IX.2.1])

$$\partial\mathcal{H}(\dot{\boldsymbol{z}}_\varepsilon(t)) \subset \partial\mathcal{H}(0) = \{\zeta \in L^2(D;\mathbb{R}^m) : \ \zeta(x) \in \partial H(0) \ \text{ for a.e. } x \in D\}, \tag{4.3.10}$$

for a.e. $t \in [0,T]$. Hence from (4.3.9) we derive (ev3)$_\varepsilon$.

Since $\mathcal{H}$ is positively homogeneous of degree one, we have $\langle \zeta, z\rangle = \mathcal{H}(z)$, for every $z \in L^2(D;\mathbb{R}^m)$ and $\zeta \in \partial\mathcal{H}(z)$. Therefore, by (4.3.9),

$$\mathcal{H}(\dot{\boldsymbol{z}}_\varepsilon(t)) = \langle \boldsymbol{\zeta}_\varepsilon(t) - \varepsilon\dot{\boldsymbol{z}}_\varepsilon(t), \dot{\boldsymbol{z}}_\varepsilon(t)\rangle, \tag{4.3.11}$$

for a.e. $t \in [0,T]$.

Choosing $\tilde{u} = \dot{\boldsymbol{v}}_\varepsilon(t) - \dot{\boldsymbol{\varphi}}(t)$ in (ev2)$_\varepsilon$ and using (4.3.11) we obtain

$$\begin{aligned} \langle \tfrac{\partial W}{\partial \theta}(\boldsymbol{z}_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t)), \dot{\boldsymbol{z}}_\varepsilon(t)\rangle + \langle \tfrac{\partial W}{\partial F}(\boldsymbol{z}_\varepsilon(t), \nabla \boldsymbol{v}_\varepsilon(t)), \nabla\dot{\boldsymbol{v}}_\varepsilon(t) - \nabla\dot{\boldsymbol{\varphi}}(t))\rangle +& \\ -\langle \boldsymbol{l}(t), \dot{\boldsymbol{v}}_\varepsilon(t) - \dot{\boldsymbol{\varphi}}(t)\rangle + \mathcal{H}(\dot{\boldsymbol{z}}_\varepsilon(t)) +& \\ +\varepsilon\|\dot{\boldsymbol{z}}_\varepsilon(t)\|_2^2 + \varepsilon\langle \nabla\dot{\boldsymbol{v}}_\varepsilon(t), \nabla\dot{\boldsymbol{v}}_\varepsilon(t) - \nabla\dot{\boldsymbol{\varphi}}(t))\rangle = 0&, \end{aligned} \tag{4.3.12}$$

for a.e. $t \in [0,T]$. By integration of (4.3.12) from 0 to t, we obtain (ev4)$_\varepsilon$.

Conversely, suppose that $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ satisfies (ev0)$_\varepsilon$, (ev1)$_\varepsilon$, (ev2)$_\varepsilon$, (ev3)$_\varepsilon$ and (ev4)$_\varepsilon$. Then, by derivation with respect to t of (ev4)$_\varepsilon$, we obtain (4.3.12), which gives (4.3.11), thanks to (ev2)$_\varepsilon$. Combining (4.3.11) with (ev3)$_\varepsilon$, it immediately follows (4.3.9) for a.e. $t \in [0,T]$, and hence $(\mathrm{ev}\hat{3})_\varepsilon$. ☐

We conclude by stating the existence theorem for the solutions of the ε-regularized problems, which will be proved in the next section.

THEOREM 4.3.6. *Let ε, $\boldsymbol{l}$, $\boldsymbol{\varphi}$, z_0, v_0, and T as in Definition 4.3.1. Then there exists a unique solution of the ε-regularized problem in the time interval $[0,T]$ with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0, v_0).*

4.4. Proof of Theorem 4.3.6

The proof is obtained via time-discretization, resolution of incremental minimum problems, and passing to the limit as the time step tends to 0.

4.4.1. The incremental minimum problem. In this section we study the incremental minimum problem used in the discrete-time formulation of the evolution problem.

Let us fix a sequence of subdivisions of $[0,T]$, $0 = t_n^0 < t_n^1 < \cdots < t_n^{k(n)} = T$, such that $\tau_n := \sup_{i=1,\dots,k(n)} \tau_n^i \to 0$, as $n \to \infty$, where $\tau_n^i := t_n^i - t_n^{i-1}$, for every $i = 1, \dots, k(n)$. We assume that $\tau_n < \frac{\varepsilon}{M}$ for every n, where M is the constant appearing in **(W.3)**.

For every $i = 0, 1, \dots, k(n)$ we set $l_n^i := \boldsymbol{l}(t_n^i)$ and $\varphi_n^i := \boldsymbol{\varphi}(t_n^i)$.

For every n, we define $(z_n^i, v_n^i) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(t_n^i)$ by induction on i: set $(z_n^0, v_n^0) := (z_0, v_0)$, and for $i > 0$ we define (z_n^i, v_n^i) as the solution (see Lemma 4.4.1 below) to the incremental minimum problem

$$\min \{\mathcal{W}(z,v) - \langle l_n^i, v\rangle + \mathcal{H}(z - z_n^{i-1}) + \tfrac{\varepsilon}{2\tau_n^i}\|z - z_n^{i-1}\|_2^2 + \tfrac{\varepsilon}{2\tau_n^i}\|\nabla v - \nabla v_n^{i-1}\|_2^2\}, \tag{4.4.1}$$

among all $z \in L^2(D;\mathbb{R}^m)$ and all $v \in \mathcal{A}(t_n^i)$.

LEMMA 4.4.1. *Let $\varepsilon > 0$, then for every n and every $i > 0$ there exists a unique solution to (4.4.1) in $L^2(D;\mathbb{R}^m) \times \mathcal{A}(t_n^i)$.*

PROOF. Let $(z_k, v_k) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(t_n^i)$ be a minimizing sequence. By the bounds on W and the assumption on $\boldsymbol{l}$, we have

$$c(\|z_k\|_2^2 + \|\nabla v_k\|_2^2) - C'(1 + \|v_k\|_{H^1}) \leq \mathcal{W}(z_k, v_k) - \langle l_n^i, v_k\rangle \leq C', \tag{4.4.2}$$

for suitable positive constants c, C'. Since $v_k \in \mathcal{A}(t_n^i)$, using Poincaré inequality we can deduce from (4.4.2) that $(z_k, v_k)_k$ is a bounded sequence in $L^2(D;\mathbb{R}^m) \times H^1(D;\mathbb{R}^N)$.

From the boundedness of the second derivative of W, guaranteed by **(W.3)**, and since $\tau_n \in (0, \varepsilon/M)$ by our assumption on τ_n, it easily follows that the functional in (4.4.1) is strictly convex, and hence weakly lower semicontinuous on $L^2(D;\mathbb{R}^m) \times H^1(D;\mathbb{R}^N)$. Now the existence of minimizers comes from the direct methods of the Calculus of Variations; the uniqueness is a consequence of the strict convexity of the functional. □

Now we derive Euler conditions for the minimizers.

THEOREM 4.4.2. *Let $\varepsilon > 0$, then for every n and every $i > 0$ we have*

$$\begin{aligned} &\mathcal{H}(\tilde{z} + z_n^i - z_n^{i-1}) - \mathcal{H}(z_n^i - z_n^{i-1}) - \langle l_n^i, \tilde{u}\rangle \geq \\ &\geq \langle -\tfrac{\partial W}{\partial \theta}(z_n^i, \nabla v_n^i), \tilde{z}\rangle - \langle \tfrac{\partial W}{\partial F}(z_n^i, \nabla v_n^i), \nabla \tilde{u}\rangle + \\ &\quad - \tfrac{\varepsilon}{\tau_n^i}\langle z_n^i - z_n^{i-1}, \tilde{z}\rangle - \tfrac{\varepsilon}{\tau_n^i}\langle \nabla v_n^i - \nabla v_n^{i-1}, \nabla\tilde{u}\rangle, \end{aligned} \tag{4.4.3}$$

for every $\tilde{z} \in L^2(D;\mathbb{R}^N)$ and $\tilde{u} \in H^1_{\Gamma_0}(0)$.

Hence we can deduce the following Euler conditions:

$$-\tfrac{\partial W}{\partial \theta}(z_n^i, \nabla v_n^i) - \tfrac{\varepsilon}{\tau_n^i}(z_n^i - z_n^{i-1}) \in \partial\mathcal{H}(z_n^i - z_n^{i-1}), \tag{4.4.4}$$

$$\langle \tfrac{\partial W}{\partial F}(z_n^i, \nabla v_n^i) + \tfrac{\varepsilon}{\tau_n^i}(\nabla v_n^i - \nabla v_n^{i-1}), \nabla\tilde{u}\rangle = \langle l_n^i, \tilde{u}\rangle \quad \textit{for every } \tilde{u} \in H^1_{\Gamma_0}(0). \tag{4.4.5}$$

Conversely, conditions (4.4.4) and (4.4.5) imply that (z_n^i, v_n^i) is a solution of (4.4.1).

PROOF. Since for every $\tilde{z} \in L^2(D;\mathbb{R}^m)$, $\tilde{u} \in H^1_{\Gamma_0}(0)$ and $s \geq 0$, $z_n^i + s\tilde{z} \in L^2(D;\mathbb{R}^m)$ and $v_n^i + s\tilde{u} \in \mathcal{A}(t_n^i)$, the minimality property of (z_n^i, v_n^i) leads to

$$\begin{aligned} &\mathcal{W}(z_n^i, v_n^i) - \langle l_n^i, v_n^i\rangle + \mathcal{H}(z_n^i - z_n^{i-1}) + \tfrac{\varepsilon}{2\tau_n^i}\|z_n^i - z_n^{i-1}\|_2^2 + \tfrac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i - \nabla v_n^{i-1}\|_2^2 \leq \\ &\leq \mathcal{W}(z_n^i + s\tilde{z}, v_n^i + s\tilde{u}) - \langle l_n^i, v_n^i + s\tilde{u}\rangle + \mathcal{H}(z_n^i + s\tilde{z} - z_n^{i-1}) + \\ &\quad + \tfrac{\varepsilon}{2\tau_n^i}\|z_n^i + s\tilde{z} - z_n^{i-1}\|_2^2 + \tfrac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i + s\nabla\tilde{u} - \nabla v_n^{i-1}\|_2^2. \end{aligned} \tag{4.4.6}$$

Hence from the convexity of $\mathcal{H}$ we can deduce for every $s \in [0,1]$

$$\begin{aligned} &s\big[\mathcal{H}(z_n^i + \tilde{z} - z_n^{i-1}) - \mathcal{H}(z_n^i - z_n^{i-1})\big] \geq \mathcal{W}(z_n^i, v_n^i) + \\ &-\mathcal{W}(z_n^i + s\tilde{z}, v_n^i + s\tilde{u}) - \langle l_n^i, v_n^i\rangle + \langle l_n^i, v_n^i + s\tilde{u}\rangle + \tfrac{\varepsilon}{2\tau_n^i}\|z_n^i - z_n^{i-1}\|_2^2 + \\ &-\tfrac{\varepsilon}{2\tau_n^i}\|z_n^i + s\tilde{z} - z_n^{i-1}\|_2^2 + \tfrac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i - \nabla v_n^{i-1}\|_2^2 - \tfrac{\varepsilon}{2\tau_n^i}\|\nabla v_n^i + s\nabla\tilde{u} - \nabla v_n^{i-1}\|_2^2. \end{aligned} \tag{4.4.7}$$

Taking the derivative of (4.4.7) with respect to s at $s = 0$ we obtain (4.4.3). For $\tilde{u} = 0$, (4.4.3) is

$$\mathcal{H}(z_n^i + \tilde{z} - z_n^{i-1}) - \mathcal{H}(z_n^i - z_n^{i-1}) \geq \langle -\tfrac{\partial W}{\partial \theta}(z_n^i, \nabla v_n^i) - \tfrac{\varepsilon}{\tau_n^i}(z_n^i - z_n^{i-1}), \tilde{z}\rangle,$$

for every $\tilde z \in L^2(D;\mathbb{R}^m)$, i.e. (4.4.4). Taking $\tilde z = 0$ in (4.4.3) we obtain (4.4.5).

Conversely, (4.4.4) and (4.4.5) imply the minimality of (z_n^i, v_n^i), thanks to the strict convexity of the functional. $\square$

REMARK 4.4.3. (4.4.4) is equivalent to

$$\tfrac{1}{\tau_n^i}(z_n^i - z_n^{i-1}) = \partial\mathcal{H}_\varepsilon^*(-\tfrac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i)). \tag{4.4.8}$$

Indeed, since $\partial\mathcal{H}$ is positively homogeneous of degree 0, (4.4.4) can be rewritten as

$$-\tfrac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i) - \tfrac{\varepsilon}{\tau_n^i}(z_n^i - z_n^{i-1}) \in \partial\mathcal{H}(\tfrac{1}{\tau_n^i}(z_n^i - z_n^{i-1})),$$

which is equivalent to

$$-\tfrac{\partial W}{\partial\theta}(z_n^i, \nabla v_n^i) \in \partial\mathcal{H}_\varepsilon(\tfrac{1}{\tau_n^i}(z_n^i - z_n^{i-1})).$$

This is equivalent to (4.4.8), thanks to a general duality formula (see, e.g., [**16**, Corollary I.5.2]).

Since $\mathcal{A}(t_n^i) = \varphi_n^i + H^1_{\Gamma_0}(0)$, for every n and every $i = 0, 1, \dots, k(n)$, there exists $u_n^i \in H^1_{\Gamma_0}(0)$ such that $v_n^i = \varphi_n^i + u_n^i$.

Set, for every n and every $i = 0, 1, \dots, k(n)$,

$$\begin{aligned} \zeta_n^i &:= -\tfrac{\partial W}{\partial\theta}(z_n^i, \nabla u_n^i + \nabla\varphi_n^i), \\ \sigma_n^i &:= \tfrac{\partial W}{\partial F}(z_n^i, \nabla u_n^i + \nabla\varphi_n^i). \end{aligned}$$

Let define the piecewise constant interpolations $(\boldsymbol{z}_n, \boldsymbol{u}_n)\colon [0,T] \to L^2(D;\mathbb{R}^m) \times H^1_{\Gamma_0}(0)$ and $(\boldsymbol{\zeta}_n, \boldsymbol{\sigma}_n)\colon [0,T] \to L^2(D;\mathbb{R}^m) \times L^2(D;\mathbb{R}^{N\times d})$ as

$$\boldsymbol{z}_n(t) := z_n^i, \quad \boldsymbol{u}_n(t) := u_n^i, \quad \boldsymbol{\zeta}_n(t) := \zeta_n^i, \quad \boldsymbol{\sigma}_n(t) := \sigma_n^i, \quad \text{for } t_n^i \le t < t_n^{i+1},$$

where we set $t_n^{k(n)+1} = T + \frac{1}{n}$. Set $\tau_n(t) := t_n^i$ whenever $t_n^i \le t < t_n^{i+1}$. We introduce also the piecewise affine interpolations $\boldsymbol{z}_n^\triangle\colon [0,T] \to L^2(D;\mathbb{R}^m)$, $\boldsymbol{u}_n^\triangle\colon [0,T] \to H^1_{\Gamma_0}(0)$, $\boldsymbol{\varphi}_n^\triangle\colon [0,T] \to H^1(D;\mathbb{R}^N)$, $\boldsymbol{l}_n^\triangle\colon [0,T] \to H^1(D;\mathbb{R}^N)^*$, defined by

$$\begin{aligned} \boldsymbol{z}_n^\triangle(t) &:= z_n^i + (t - t_n^i)\tfrac{z_n^{i+1} - z_n^i}{t_n^{i+1} - t_n^i}, \\ \boldsymbol{u}_n^\triangle(t) &:= u_n^i + (t - t_n^i)\tfrac{u_n^{i+1} - u_n^i}{t_n^{i+1} - t_n^i}, \\ \boldsymbol{\varphi}_n^\triangle(t) &:= \varphi_n^i + (t - t_n^i)\tfrac{\varphi_n^{i+1} - \varphi_n^i}{t_n^{i+1} - t_n^i}, \\ \boldsymbol{l}_n^\triangle(t) &:= l_n^i + (t - t_n^i)\tfrac{l_n^{i+1} - l_n^i}{t_n^{i+1} - t_n^i} \end{aligned} \tag{4.4.9}$$

for $t_n^i \le t \le t_n^{i+1}$.

Observe that, thanks to Remark 4.4.3 and to (1.1.5), we can obtain from (4.4.4) that

$$\tfrac{z_n^i - z_n^{i-1}}{t_n^i - t_n^{i-1}} = N_K^\varepsilon(\zeta_n^i). \tag{4.4.10}$$

Analogously we can deduce from (4.4.5) that

$$\langle \sigma_n^i + \varepsilon(\tfrac{\nabla u_n^i - \nabla u_n^{i-1}}{t_n^i - t_n^{i+1}} + \tfrac{\nabla\varphi_n^i - \varphi_n^{i-1}}{t_n^i - t_n^{i-1}}), \nabla\tilde u\rangle = \langle l_n^{i-1}, \tilde u\rangle, \tag{4.4.11}$$

for every $\tilde u \in H^1_{\Gamma_0}(0)$.

4.4.2. A priori estimates. Now we obtain an a priori bound on the piecewise constant interpolations, from an energy estimate for the solutions of the incremental problems.

Since $u_n^{i-1} + \varphi_n^i \in \mathcal{A}(t_n^i)$, from the minimum property of $(z_n^i, u_n^i + \varphi_n^i)$ we deduce the following inequality:

$$\begin{aligned} &\mathcal{W}(z_n^i, u_n^i + \varphi_n^i) - \langle l_n^i, u_n^i + \varphi_n^i \rangle + \mathcal{H}(z_i^n - z_n^{i-1}) + \\ &+ \tfrac{\varepsilon}{2\tau_n^i}\|z_n^i - z_n^{i-1}\|_2^2 + \tfrac{\varepsilon}{2\tau_n^i}\|\nabla u_n^i - \nabla u_n^{i-1} + \nabla\varphi_n^i - \nabla\varphi_n^{i-1}\|_2^2 \le \\ &\le \mathcal{W}(z_n^{i-1}, u_n^{i-1} + \varphi_n^{i-1}) - \langle l_n^{i-1}, u_n^{i-1} + \varphi_n^{i-1} \rangle + \\ &+ \tfrac{\varepsilon}{2\tau_n^i}\|\nabla\varphi_n^i - \nabla\varphi_n^{i-1}\|_2^2 - \int_{t_n^{i-1}}^{t_n^i} \langle \dot{\boldsymbol{l}}(s), u_n^{i-1} + \boldsymbol{\varphi}(s) \rangle \, ds + \\ &- \int_{t_n^{i-1}}^{t_n^i} \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle \, ds + \int_{t_n^{i-1}}^{t_n^i} \langle \tfrac{\partial W}{\partial F}(z_n^{i-1}, \nabla u_n^{i-1} + \nabla\boldsymbol{\varphi}(s)), \nabla\dot{\boldsymbol{\varphi}}(s) \rangle \, ds. \end{aligned} \tag{4.4.12}$$

Fixed $t \in [0,T]$, iterating (4.4.12) we obtain

$$\begin{aligned} &\mathcal{W}(\boldsymbol{z}_n(t), \boldsymbol{u}_n(t) + \boldsymbol{\varphi}(\tau_n(t))) - \langle \boldsymbol{l}(\tau_n(t)), \boldsymbol{u}_n(t) + \boldsymbol{\varphi}(\tau_n(t)) \rangle + \mathrm{Var}_H(\boldsymbol{z}_n; 0, t) + \\ &+ \tfrac{\varepsilon}{2}\int_0^{\tau_n(t)} \|\dot{\boldsymbol{z}}_n^{\triangle}(s)\|_2^2 \, ds + \tfrac{\varepsilon}{4}\int_0^{\tau_n(t)} \|\nabla\dot{\boldsymbol{u}}_n^{\triangle}(s)\|_2^2 \, ds \le \\ &\le \mathcal{W}(z_0, v_0) - \langle \boldsymbol{l}(0), v_0 \rangle + \varepsilon \int_0^{\tau_n(t)} \|\nabla\dot{\boldsymbol{\varphi}}(s)\|_2^2 \, ds + \\ &+ \int_0^{\tau_n(t)} \langle \tfrac{\partial W}{\partial F}(\boldsymbol{z}_n(s), \nabla\boldsymbol{u}_n(s) + \nabla\boldsymbol{\varphi}(s)), \nabla\dot{\boldsymbol{\varphi}}(s) \rangle \, ds + \\ &- \int_0^{\tau_n(t)} [\langle \dot{\boldsymbol{l}}(s), u_n(s) + \boldsymbol{\varphi}(s) \rangle + \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle] \, ds, \end{aligned} \tag{4.4.13}$$

where we have used the identity

$$\tfrac{\varepsilon}{\tau_n^i}\|\nabla\varphi_n^i - \nabla\varphi_n^{i-1}\|_2^2 = \tfrac{\varepsilon}{\tau_n^i}\|\int_{t_n^{i-1}}^{t_n^i} \nabla\dot{\boldsymbol{\varphi}}(t) \, dt\|_2^2, \tag{4.4.14}$$

for every $i = 1, \ldots, k(n)$.

Using the fact that $\sup_{t\in[0,T]} \|\boldsymbol{l}(t)\|_{(H^1)^*}$, $\sup_{t\in[0,T]} \|\nabla\boldsymbol{\varphi}(t)\|_2$, $\int_0^T \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*} \, dt$ and $\int_0^T \|\nabla\dot{\boldsymbol{\varphi}}(t)\|_2 \, dt$ are bounded, the growth hypothesis on W, **(W.1)**, and the fact that $\boldsymbol{z}_n \in L^\infty([0,T]; L^2(D;\mathbb{R}^m))$, $\boldsymbol{u}_n \in L^\infty([0,T]; H^1(D;\mathbb{R}^N))$ (since they are piecewise constant functions), (4.4.13) leads to

$$\tilde{c}(\|\boldsymbol{z}_n(t)\|_2 + \|\nabla\boldsymbol{u}_n(t)\|_2)^2 \le \tilde{C} \sup_{s\in[0,T]} (1 + \|\boldsymbol{z}_n(s)\|_2 + \|\nabla\boldsymbol{u}_n(s)\|_2),$$

for suitable positive constants $\tilde{c}, \tilde{C}$. Since this can be repeated for every $t \in [0,T]$, we can conclude that there exists a positive constant C_ε, depending on ε but independent of n, such that

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_n(t)\|_2 \le C_\varepsilon, \qquad \mathrm{Var}_H(\boldsymbol{z}_n;0,T) \le C_\varepsilon, \qquad \int_0^T \|\dot{\boldsymbol{z}}_n^{\triangle}(t)\|_2^2\,dt \le C_\varepsilon; \tag{4.4.15}$$

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{u}_n(t)\|_2 \le C_\varepsilon, \qquad \int_0^T \|\nabla \dot{\boldsymbol{u}}_n^{\triangle}(t)\|_2^2\,dt \le C_\varepsilon, \tag{4.4.16}$$

for every $t \in [0,T]$.

4.4.3. Passage to the limit. To establish the convergence of the interpolations we need the following Lemma, based on Gronwall's inequality.

Lemma 4.4.4. *The sequences $(\boldsymbol{z}_n)_n$, $(\boldsymbol{u}_n)_n$ satisfy*

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_n(t) - \boldsymbol{z}_m(t)\|_2 \to 0,$$
$$\sup_{t\in[0,T]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t)\|_{H^1} \to 0,$$

as $n, m \to \infty$.

Proof. From the construction of z_n^i, u_n^i, and using (4.4.10), (4.4.11), we can deduce that, for every $\tilde z \in L^2(D;\mathbb{R}^N)$, $\tilde u \in H^1_{\Gamma_0}(0)$,

$$\begin{aligned}\langle \nabla u_n^j + \nabla \varphi_n^j, \nabla \tilde u\rangle - \langle \nabla u_n^{j-1} + \nabla\varphi_n^{j-1}, \nabla\tilde u\rangle + \langle z_n^j, \tilde z\rangle - \langle z_n^{j-1}, \tilde z\rangle =\\ = -\tfrac{\tau_n^j}{\varepsilon}[\langle \sigma_n^j, \nabla\tilde u\rangle - \langle l_n^j, \tilde u\rangle] + \tau_n^j \langle N_K^\varepsilon(\zeta_n^j), \tilde z\rangle.\end{aligned}$$

Fixed $t\in[0,T]$, for every n, there exists i such that $t_n^i \le t < t_n^{i+1}$; summing for j from 1 to i we obtain

$$\begin{aligned}&\langle \boldsymbol{z}_n(t), \tilde z\rangle - \langle z_0, \tilde z\rangle + \langle \nabla \boldsymbol{u}_n(t) + \nabla\boldsymbol{\varphi}(\tau_n(t)), \nabla\tilde u\rangle - \langle \nabla u_0 + \nabla\boldsymbol{\varphi}(0), \nabla\tilde u\rangle =\\
&= \tfrac{1}{\varepsilon}\sum_{j=1}^i \tau_n^j \left[\varepsilon\,\langle N_K^\varepsilon(\zeta_n^{j-1}), \tilde z\rangle - \langle \sigma_n^{j-1}, \tilde u\rangle + \langle l_n^{j-1}, \tilde u\rangle + \right.\\
&\left. + \ \varepsilon\langle N_K^\varepsilon(\zeta_n^j) - N_K^\varepsilon(\zeta_n^{j-1}), \tilde z\rangle - \langle \sigma_n^j - \sigma_n^{j-1}, \nabla\tilde u\rangle + \langle l_n^j - l_n^{j-1}, \tilde u\rangle\right] =\\
&= \tfrac{1}{\varepsilon}\int_0^t \left[\varepsilon\,\langle N_K^\varepsilon(\boldsymbol{\zeta}_n(s)), \tilde z\rangle - \langle \boldsymbol{\sigma}_n(s), \nabla\tilde u\rangle + \langle \boldsymbol{l}(\tau_n(s), \tilde u\rangle\right] ds + R_n(t),\end{aligned} \tag{4.4.17}$$

where

$$\begin{aligned}R_n(t) :=& -\tfrac{1}{\varepsilon}\int_{\tau_n(t)}^t \left[\varepsilon\,\langle N_K^\varepsilon(\boldsymbol{\zeta}_n(s)), \tilde z\rangle - \langle \boldsymbol{\sigma}_n(s), \nabla\tilde u\rangle + \langle \boldsymbol{l}(\tau_n(s)), \tilde u\rangle\right] ds +\\
&+ \tfrac{1}{\varepsilon}\sum_{j=1}^i \tau_n^j\big[\,\varepsilon\langle N_K^\varepsilon(\zeta_n^j) - N_K^\varepsilon(\zeta_n^{j-1}), \tilde z\rangle - \langle \sigma_n^j - \sigma_n^{j-1}, \nabla\tilde u\rangle + \langle l_n^j - l_n^{j-1}, \tilde u\rangle\big].\end{aligned}$$

Observe that, since $\frac{\partial W}{\partial\theta}$, $\frac{\partial W}{\partial F}$ are M-Lipschitz thanks to **(W.3)**,

$$\|\boldsymbol{\zeta}_n(s)\|_2 \le \tilde M(1 + \|\boldsymbol{z}_n(s)\|_2 + \|\nabla\boldsymbol{u}_n(s) + \nabla\boldsymbol{\varphi}(\tau_n(s))\|_2), \tag{4.4.18}$$

$$\|\boldsymbol{\sigma}_n(s)\|_2 \le \tilde M(1 + \|\boldsymbol{z}_n(s)\|_2 + \|\nabla\boldsymbol{u}_n(s) + \nabla\boldsymbol{\varphi}(\tau_n(s))\|_2), \tag{4.4.19}$$

for a suitable positive constant $\tilde{M}$ and for every $s \in [0,T]$. Hence from (4.4.15) and (4.4.16), we can deduce that

$$\int_{\tau_n(t)}^t \|\boldsymbol{\zeta}_n(s)\|_2 \, ds \leq \tilde{M}\tilde{C}_\varepsilon \tau_n, \tag{4.4.20}$$

$$\int_{\tau_n(t)}^t \|\boldsymbol{\sigma}_n(s)\|_2 \, ds \leq \tilde{M}\tilde{C}_\varepsilon \tau_n. \tag{4.4.21}$$

Since N_K^ε is $1/\varepsilon$-Lipschitz, and $\partial W/\partial\theta$, $\partial W/\partial F$ are M-Lipschitz, thanks to (4.4.20) and (4.4.21) we can estimate $R_n(t)$ in the following way

$$|R_n(t)| \leq \tfrac{1}{\varepsilon}\beta_\varepsilon \tau_n (\|\tilde{z}\|_2 + \|\nabla \tilde{u}\|_2), \tag{4.4.22}$$

for a suitable positive constant β_ε, depending on ε but independent of t and n.

Let n, m be two different indexes. Subtracting term by term the equations corresponding to (4.4.17), we obtain, for every $\tilde{z} \in L^2(D;\mathbb{R}^m)$ and $\tilde{u} \in H^1_{\Gamma_0}(0)$,

$$\begin{aligned}
&\langle \boldsymbol{z}_n(t) - \boldsymbol{z}_m(t), \tilde{z}\rangle + \langle \nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t) + \nabla\boldsymbol{\varphi}(\tau_n(t)) - \nabla\boldsymbol{\varphi}(\tau_m(t)), \nabla\tilde{u}\rangle = \\
&= \tfrac{1}{\varepsilon}\int_0^t \big[\, \varepsilon\langle N_K^\varepsilon(\boldsymbol{\zeta}_n(s)) - N_K^\varepsilon(\boldsymbol{\zeta}_m(s)), \tilde{z}\rangle - \langle \boldsymbol{\sigma}_n(s) - \boldsymbol{\sigma}_m(s), \nabla\tilde{u}\rangle\big]\, ds + \\
&\quad + \int_0^t \langle \boldsymbol{l}(\tau_n(s)) - \boldsymbol{l}(\tau_m(s)), \tilde{u}\rangle\, ds + R_n(t) - R_m(t).
\end{aligned} \tag{4.4.23}$$

Now using again the fact that N_K^ε, $\partial W/\partial\theta$ and $\partial W/\partial F$ are Lipschitzian, and the estimate (4.4.22), we can deduce that

$$\begin{aligned}
&\langle \boldsymbol{z}_n(t) - \boldsymbol{z}_m(t), \tilde{z}\rangle + \langle \nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t) + \nabla\boldsymbol{\varphi}(\tau_n(t)) - \nabla\boldsymbol{\varphi}(\tau_m(t)), \nabla\tilde{u}\rangle \leq \\
&\leq \tfrac{\gamma_\varepsilon}{\varepsilon}\Big\{ \int_0^t \big[\|\boldsymbol{z}_n(s) - \boldsymbol{z}_m(s)\|_2 - \|\nabla \boldsymbol{u}_n(s) - \nabla\boldsymbol{u}_m(s)\|_2 + \\
&\quad + \|\nabla\boldsymbol{\varphi}(\tau_n(s)) - \nabla\boldsymbol{\varphi}(\tau_m(s))\|_2 + \|\boldsymbol{l}_n(s) - \boldsymbol{l}_m(s)\|_{(H^1)^*}\big]\, ds + \\
&\quad + \beta_\varepsilon(\tau_n + \tau_m)\Big\}(\|\tilde{z}\|_2 + \|\nabla\tilde{u}\|_2),
\end{aligned} \tag{4.4.24}$$

for a suitable positive constant γ_ε.

Since $\boldsymbol{\varphi} \in H^1([0,T];H^1(D;\mathbb{R}^N))$ and $\boldsymbol{l} \in H^1([0,T];H^1(D;\mathbb{R}^N)^*)$, there exists a positive constant α, such that

$$\|\nabla\boldsymbol{\varphi}(t_1) - \nabla\boldsymbol{\varphi}(t_2)\|_2 \leq \alpha|t_1 - t_2|^{1/2}, \tag{4.4.25}$$

$$\|\boldsymbol{l}(t_1) - \boldsymbol{l}(t_2)\|_{(H^1)^*} \leq \alpha|t_1 - t_2|^{1/2}, \tag{4.4.26}$$

for every $t_1, t_2 \in [0,T]$.

It follows that

$$\|\nabla\boldsymbol{\varphi}(\tau_n(t)) - \nabla\boldsymbol{\varphi}(\tau_m(t))\|_2 \leq \alpha((\tau_n)^{1/2} + (\tau_m)^{1/2}) \tag{4.4.27}$$

$$\|\boldsymbol{l}(\tau_n(t)) - \boldsymbol{l}(\tau_m(t)\|_{(H^1)^*} \leq \alpha((\tau_n)^{1/2} + (\tau_m)^{1/2}), \tag{4.4.28}$$

for every $t \in [0,T]$ and every n, m.

If we choose $\tilde{z} = \boldsymbol{z}_n(t) - \boldsymbol{z}_m(t)$ and $\tilde{u} = \boldsymbol{u}_n(t) - \boldsymbol{u}_m(t)$, taking into account (4.4.27) and (4.4.28), (4.4.24) gives

$$\|\boldsymbol{z}_n(t) - \boldsymbol{z}_m(t)\|_2 + \|\nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t)\|_2 \leq$$
$$= \tfrac{\gamma_\varepsilon}{\varepsilon} \int_0^t \left[\|\boldsymbol{z}_n(s) - \boldsymbol{z}_m(s)\|_2 + \|\nabla \boldsymbol{u}_n(s) - \nabla \boldsymbol{u}_m(s)\|_2\right] \, ds + \tilde{\alpha}((\tau_n)^{1/2} + (\tau_m)^{1/2}),$$

for a suitable positive constant $\tilde{\alpha}$ independent of t, m and n.

Applying Gronwall's inequality we conclude that

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_n(t) - \boldsymbol{z}_m(t)\|_2 \to 0,$$
$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{u}_n(t) - \nabla \boldsymbol{u}_m(t)\|_2 \to 0,$$

for n, m tending to ∞. Since $\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t) \in H^1_{\Gamma_0}(0)$, applying Poincaré inequality we obtain

$$\sup_{t\in[0,T]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}_m(t)\|_{H^1} \to 0 \tag{4.4.29}$$

as n, m tend to ∞. □

From Lemma (4.4.4), we can deduce that there exist

$$\boldsymbol{z}\colon [0,T] \to L^2(D; \mathbb{R}^m),$$
$$\boldsymbol{u}\colon [0,T] \to H^1(D; \mathbb{R}^N),$$

bounded, such that

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_n(t) - \boldsymbol{z}(t)\|_2 \to 0, \tag{4.4.30}$$
$$\sup_{t\in[0,T]} \|\boldsymbol{u}_n(t) - \boldsymbol{u}(t)\|_{H^1} \to 0. \tag{4.4.31}$$

Moreover $\boldsymbol{u}(t) \in H^1_{\Gamma_0}(0)$, for every $t \in [0,T]$.

Set

$$\boldsymbol{\zeta}(t) := -\tfrac{\partial W}{\partial \theta}(\boldsymbol{z}(t), \nabla \boldsymbol{u}(t) + \nabla \boldsymbol{\varphi}(t)) \tag{4.4.32}$$
$$\boldsymbol{\sigma}(t) := \tfrac{\partial W}{\partial F}(\boldsymbol{z}(t), \nabla \boldsymbol{u}(t) + \nabla \boldsymbol{\varphi}(t)). \tag{4.4.33}$$

Thanks to (4.4.25) and to the convergence of $\boldsymbol{z}_n$, $\boldsymbol{u}_n$, we have

$$\sup_{t\in[0,T]} \|\boldsymbol{\zeta}_n(t) - \boldsymbol{\zeta}(t)\|_2 \to 0, \tag{4.4.34}$$
$$\sup_{t\in[0,T]} \|\boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t)\|_2 \to 0, \tag{4.4.35}$$

as n tends to ∞.

Thanks to (4.4.15) and (4.4.16), we have that $(\boldsymbol{z}_n^{\triangle})_n$ and $(\boldsymbol{u}_n^{\triangle})_n$ are bounded sequences in $H^1([0,T]; L^2(D; \mathbb{R}^m))$ and $H^1([0,T]; H^1(D; \mathbb{R}^N))$, respectively; hence there exist $\hat{\boldsymbol{z}}$, $\hat{\boldsymbol{u}}$ such that, up to subsequences, $\boldsymbol{z}_n^{\triangle} \rightharpoonup \hat{\boldsymbol{z}}$ and $\boldsymbol{u}_n^{\triangle} \rightharpoonup \hat{\boldsymbol{u}}$ weakly in $H^1([0,T]; L^2(D; \mathbb{R}^m))$ and $H^1([0,T]; H^1(D; \mathbb{R}^N))$, respectively.

Moreover using the identities

$$\boldsymbol{z}_n^{\triangle}(t) = \boldsymbol{z}_n(t) + \int_{\tau_n(t)}^{t} \dot{\boldsymbol{z}}_n^{\triangle}(s)\ ds,$$

$$\nabla \boldsymbol{u}_n^{\triangle}(t) = \nabla \boldsymbol{u}_n(t) + \int_{\tau_n(t))}^{t} \nabla \dot{\boldsymbol{u}}_n^{\triangle}(s)\ ds,$$

for every $t \in [0,T]$, we deduce that

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_n^{\triangle}(t) - \boldsymbol{z}_n(t)\|_2 \to 0,$$

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{u}_n^{\triangle}(t) - \nabla \boldsymbol{u}_n(t)\|_2 \to 0.$$

Hence we can conclude that $\hat{\boldsymbol{z}} = \boldsymbol{z}$, $\nabla \hat{\boldsymbol{u}} = \nabla \boldsymbol{u}$ and the whole sequences $\boldsymbol{z}_n^{\triangle}$ and $\boldsymbol{u}_n^{\triangle}$ satisfy

$$\boldsymbol{z}_n^{\triangle} \rightharpoonup \boldsymbol{z} \quad \text{weakly in } H^1([0,T]; L^2(D;\mathbb{R}^m)), \tag{4.4.36}$$

$$\boldsymbol{u}_n^{\triangle} \rightharpoonup \boldsymbol{u} \quad \text{weakly in } H^1([0,T]; H^1(D;\mathbb{R}^N)). \tag{4.4.37}$$

It is immediate to see that $(\mathrm{ev0})_\varepsilon$ follows from the construction of $(\boldsymbol{z}_n, \boldsymbol{v}_n)$ and from (4.4.30), (4.4.31).

Since $\boldsymbol{u}(t) \in H^1_{\Gamma_0}(0)$ for every $t \in [0,T]$ also $(\mathrm{ev1})_\varepsilon$ is immediate.

We prove now $(\mathrm{ev2})_\varepsilon$. From the construction of $\boldsymbol{u}_n^{\triangle}$ and (4.4.11), it follows that

$$\langle \boldsymbol{\sigma}_n(t) + R_n^u(t) + \varepsilon(\nabla \dot{\boldsymbol{u}}_n^{\triangle}(t) + \nabla \dot{\boldsymbol{\varphi}}_n^{\triangle}(t)), \nabla \tilde{u}\rangle = \langle \boldsymbol{l}(\tau_n(t)) + R_n^l(t), \tilde{u}\rangle, \tag{4.4.38}$$

for every $\tilde{u} \in H^1_{\Gamma_0}(0)$, where $R_n^u(t) := \sigma_n^{i+1} - \sigma_n^i$ and $R_n^l(t) := l_n^{i+1} - l_n^i$, for $t_n^i < t < t_n^{i+1}$.

Thanks to (4.4.26),

$$\sup_{t\in[0,T]} \|R_n^l(t)\|_{(H^1)^*} \to 0.$$

Using the fact that N_K^ε is $1/\varepsilon$-Lipschitz, **(W.3)**, the hypothesis on $\boldsymbol{\varphi}$, and (4.4.15), (4.4.16), we deduce that

$$\sup_{t\in[0,T]} \|R_n^u(t)\|_2 \to 0.$$

From (4.4.14) we deduce that $\int_0^T \|\nabla \dot{\boldsymbol{\varphi}}_n^{\triangle}(t)\|_2^2\ dt$ is uniformly bounded with respect to n and then $\nabla \dot{\boldsymbol{\varphi}}_n^{\triangle} \rightharpoonup \nabla \dot{\boldsymbol{\varphi}}$ weakly in $L^2([0,T]; L^2(D;\mathbb{R}^{N\times d}))$.

Thus, $t \mapsto \boldsymbol{\sigma}_n(t) + R_n^u(t) + \varepsilon(\nabla \dot{\boldsymbol{u}}_n^{\triangle}(t) + \nabla \dot{\boldsymbol{\varphi}}_n^{\triangle}(t))$ weakly converges in $L^2([0,T]; L^2(D;\mathbb{R}^{N\times d})$ to $t \mapsto \boldsymbol{\sigma}(t) + \varepsilon(\nabla \dot{\boldsymbol{u}}(t) + \nabla \dot{\boldsymbol{\varphi}}(t))$; $t \mapsto \boldsymbol{l}(\tau_n(t)) + R_n^l(t)$ strongly converges in $L^2([0,T]; H^1(D;\mathbb{R}^N)^*)$ to $\boldsymbol{l}(t)$ as $n \to +\infty$.

Therefore from (4.4.38) we can obtain $(\mathrm{ev2})_\varepsilon$.

Finally we prove $(\mathrm{ev3})_\varepsilon$. From the construction of $\boldsymbol{z}_n^{\triangle}$ and (4.4.10), it follows that

$$\dot{\boldsymbol{z}}_n^{\triangle}(t) = N_K^\varepsilon(\boldsymbol{\zeta}_n(t)) + R_n^z(t) \quad \text{a.e. in } D,$$

where $R_n^z(t) := N_K^\varepsilon(\zeta_n^{i+1}) - N_K^\varepsilon(\zeta_n^i)$, for $t_n^i < t < t_n^{i+1}$.

Repeating the previous argument we deduce that $\sup_{t\in[0,T]} \|R_n^z(t)\|_2 \to 0$, as $n \to +\infty$, so that, taking into account (4.4.34), we conclude that

$$\sup_{t\in[0,T]} \|\dot{\boldsymbol{z}}_n^{\triangle}(t) - N_K^\varepsilon(\boldsymbol{\zeta}(t))\|_2 \to 0. \tag{4.4.39}$$

In particular this implies that $\dot{\boldsymbol{z}}_n^{\triangle}$ converges strongly in $L^\infty([0,T];L^2(D;\mathbb{R}^m))$ and the limit must coincide with $\dot{\boldsymbol{z}}$, thanks to (4.4.36); hence from (4.4.39) we obtain

$$\dot{\boldsymbol{z}}(t) = N_K^\varepsilon(\boldsymbol{\zeta}(t)), \quad \text{a.e. in} \quad D,$$

for a.e. $t \in [0,T]$.

4.4.4. Uniqueness. It remains to show that the solution of the ε-regularized problem is unique.

Let $(\boldsymbol{z}_1, \boldsymbol{v}_1)$, $(\boldsymbol{z}_2, \boldsymbol{v}_2)$ be two solutions of the ε-regularized problem in the time interval $[0,T]$ with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$, and initial condition (z_0, v_0), and set

$$\begin{aligned} \boldsymbol{\zeta}_i(t) &:= -\tfrac{\partial W}{\partial \theta}(\boldsymbol{z}_i(t), \nabla \boldsymbol{v}_i(t)), \\ \boldsymbol{\sigma}_i(t) &:= \tfrac{\partial W}{\partial F}(\boldsymbol{z}_i(t), \nabla \boldsymbol{v}_i(t)), \end{aligned}$$

for $i = 1,2$.

In particular the following equations hold for a.e. $t \in [0,T]$:

$$\begin{aligned} \dot{\boldsymbol{z}}_i(t) &= N_K^\varepsilon(\boldsymbol{\zeta}_i(t)) \quad \text{a.e. in } D, \\ \langle \boldsymbol{\sigma}_i(t) + \varepsilon \nabla \dot{\boldsymbol{v}}_i(t), \nabla \tilde{u} \rangle &= \langle \boldsymbol{l}(t), \tilde{u} \rangle \quad \text{for every } \tilde{u} \in H^1_{\Gamma_0}(0), \end{aligned}$$

for $i = 1,2$.

Hence, for a.e. $t \in [0,T]$, for every $\tilde{z} \in L^2(D;\mathbb{R}^m)$, and every $\tilde{u} \in H^1_{\Gamma_0}(0)$, we have

$$\begin{aligned} &\langle \nabla \dot{\boldsymbol{v}}_1(t) - \nabla \dot{\boldsymbol{v}}_2(t), \nabla \tilde{u} \rangle + \langle \dot{\boldsymbol{z}}_1(t) - \dot{\boldsymbol{z}}_2(t), \tilde{z} \rangle = \\ &= -\tfrac{1}{\varepsilon} \langle \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t), \nabla \tilde{u} \rangle + \langle N_K^\varepsilon(\boldsymbol{\zeta}_1(t)) - N_K^\varepsilon(\boldsymbol{\zeta}_2(t)), \tilde{z} \rangle. \end{aligned}$$

Therefore, by integration and $(\text{ev0})_\varepsilon$, we obtain

$$\begin{aligned} &\langle \nabla \boldsymbol{v}_1(t) - \nabla \boldsymbol{v}_2(t), \nabla \tilde{u} \rangle + \langle \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t), \tilde{z} \rangle = \\ &\int_0^t \left[-\tfrac{1}{\varepsilon} \langle \boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s), \nabla \tilde{u} \rangle + \langle N_K^\varepsilon(\boldsymbol{\zeta}_1(s)) - N_K^\varepsilon(\boldsymbol{\zeta}_2(s)), \tilde{z} \rangle \right] ds. \end{aligned} \tag{4.4.40}$$

We observe that $\boldsymbol{v}_1(t) - \boldsymbol{v}_2(t) \in H^1_{\Gamma_0}(0)$, for every $t \in [0,T]$. Hence we can take $\tilde{z} = \boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)$, $\tilde{u} = \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)$, and we derive from (4.4.40) the following estimate:

$$\begin{aligned} &\|\boldsymbol{z}_1(t) - \boldsymbol{z}_2(t)\|_2 + \|\nabla \boldsymbol{v}_1(t) - \nabla \boldsymbol{v}_2(t)\|_2 \le \\ &\tfrac{M'}{\varepsilon} \int_0^t \left[(\|\boldsymbol{z}_1(s) - \boldsymbol{z}_2(s)\|_2 + \|\nabla \boldsymbol{v}_1(s) - \nabla \boldsymbol{v}_2(s)\|_2) \right] ds, \end{aligned}$$

for a suitable positive constant M' and for a.e. $t \in [0,T]$.

Hence Gronwall's inequality guarantees that $\boldsymbol{z}_1(t) = \boldsymbol{z}_2(t)$ and $\boldsymbol{v}_1(t) = \boldsymbol{v}_2(t)$, for a.e. $t \in [0,T]$; since, for $i = 1,2$, $\boldsymbol{z}_i$ and $\boldsymbol{v}_i$ are absolutely continuous functions from $[0,T]$ into $L^2(D;\mathbb{R}^m)$ and $H^1(D;\mathbb{R}^N)$, respectively, we have the thesis.

4.5. Some properties of the solutions of the regularized problems

In this section we want to point out some useful properties satisfied by the solutions of the ε-regularized problems.

Remark 4.5.1. In the special case of $\boldsymbol{l} \equiv 0$, $\Gamma_0 = \partial\Omega$, $\boldsymbol{\varphi}(t,x) = F(t)x$, for $F(t) \in H^1([0,T];\mathbb{R}^{N\times d})$, for every $t \in [0,T]$ and a.e. $x \in D$, and $v_0(x) = F(0)x$, $z_0 \equiv \theta_0 \in \mathbb{R}^m$, the solution $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ of the ε-regularized problems satisfies the following properties:

(1) $\boldsymbol{v}_\varepsilon = \boldsymbol{\varphi}$
(2) $x \mapsto \boldsymbol{z}_\varepsilon(t,x)$ is a.e. constant on D, for a.e. $t \in [0,T]$.

Indeed the Cauchy problem

$$\begin{cases} \dot{\theta}_\varepsilon(t) = N_K^\varepsilon(-\frac{\partial W}{\partial \theta}(\theta_\varepsilon(t), F(t))) \\ \theta_\varepsilon(0) = \theta_0 \end{cases}$$

has a unique solution $\theta_\varepsilon \colon [0,T] \to \mathbb{R}^m$, since the right hand side is Lipschitz.

The function $(\boldsymbol{z}_\varepsilon(t), \boldsymbol{v}_\varepsilon(t)) = (\theta_\varepsilon(t), \boldsymbol{\varphi}(t))$ satisfies conditions (ev0)$_\varepsilon$, (ev1)$_\varepsilon$, (ev2)$_\varepsilon$, (ev3)$_\varepsilon$, and (ev4)$_\varepsilon$, hence by uniqueness it is the solution of the ε-regularized problem.

Using the energy equality, we can prove the following bounds on the solution of the ε-regularized problems.

LEMMA 4.5.2. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, z_0, v_0, and $T > 0$ be as in Definition (4.3.1). Then there exists a positive constant C', independent of ε, such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{z}_\varepsilon(t)\|_2 \leq C', \quad \mathrm{Var}_H(\boldsymbol{z}_\varepsilon; 0, T) \leq C', \quad \varepsilon \int_0^T \|\dot{\boldsymbol{z}}_\varepsilon(s)\|_2^2 \, ds \leq C', \tag{4.5.1}$$

$$\sup_{t\in[0,T]} \|\nabla \boldsymbol{v}_\varepsilon(t)\|_2 \leq C', \quad \varepsilon \int_0^T \|\nabla \dot{\boldsymbol{v}}_\varepsilon(s)\|_2^2 \, ds \leq C'. \tag{4.5.2}$$

PROOF. The proof can be obtained from the energy equality for $(\boldsymbol{z}_\varepsilon, \boldsymbol{v}_\varepsilon)$ reasoning as in the second step of the proof of Theorem 4.3.6.

□

REMARK 4.5.3. From (4.5.1) and (4.5.2) we can deduce that, for every sequence $\varepsilon_k \to 0$, we have $\varepsilon_k \dot{\boldsymbol{z}}_{\varepsilon_k} \to 0$ and $\varepsilon_k \nabla \dot{\boldsymbol{v}}_{\varepsilon_k} \to 0$ strongly in $L^2([0,T]; L^2(D;\mathbb{R}^m))$ and $L^2([0,T]; L^2(D;\mathbb{R}^{N\times d}))$, respectively. In particular

$$\varepsilon_k \dot{\boldsymbol{z}}_{\varepsilon_k}(t) \to 0 \quad \text{strongly in} \quad L^2(D;\mathbb{R}^m), \tag{4.5.3}$$

$$\varepsilon_k \nabla \dot{\boldsymbol{v}}_{\varepsilon_k}(t) \to 0 \quad \text{strongly in} \quad L^2(D;\mathbb{R}^N), \tag{4.5.4}$$

for a.e. $t \in [0,T]$.

4.6. Approximable quasistatic evolution

In this section we give the definition of approximable quasistatic evolution in terms both of stochastic processes and of compatible systems of Young measures. We prove an existence result and that this evolution satisfies suitable properties of equilibrium, dual constraint and an energy inequality, so that it can be considered as a solution of our evolution problem.

4.6.1. Approximable quasistatic evolution in terms of stochastic processes. Here we give the definition using a probabilistic language.

DEFINITION 4.6.1. Given a boundary datum $\boldsymbol{\varphi} \in H^1([0,T]; H^1(D;\mathbb{R}^N))$, an external load $\boldsymbol{l} \in H^1([0,T]; H^1(D;\mathbb{R}^N)^*)$, an initial condition $(z_0, v_0) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(0)$ satisfying (4.3.1) and (4.3.2), and $T > 0$, an *approximable quasistatic evolution* of stochastic processes in the time interval $[0,T]$ is a pair of stochastic processes $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ on a $(D, \mathcal{L}^d)$-probability space $(D \times \Omega, \mathcal{B}(D) \otimes \mathcal{F}, P)$, with $\boldsymbol{Z}_t \in L^2(D \times \Omega; \mathbb{R}^m)$ and 2-weakly* left continuous and $\boldsymbol{Y}_t \in L^2(D \times \Omega; \mathbb{R}^{N\times d})$, for which there exist a positive sequence $\varepsilon_k \to 0$ and a subset Θ of $[0,T]$ with $0 \in \Theta$ and $\mathcal{L}^1([0,T]\setminus\Theta) = 0$, such that the solutions $(\boldsymbol{z}_{\varepsilon_k}, \boldsymbol{v}_{\varepsilon_k})$ of the ε_k-regularized problems satisfy the following conditions:

(a) for every finite sequence $t_1 < \cdots < t_n$ in Θ, we have

$$(\pi_D, \boldsymbol{z}_{\varepsilon_k}(t_1), \ldots, \boldsymbol{z}_{\varepsilon_k}(t_n))(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_{t_1}, \ldots, \boldsymbol{Z}_{t_n})(P) \quad \text{2-weakly*}$$

as $k \to \infty$;

(b) for every $t \in \Theta$, there exists a subsequence $(\varepsilon_{k_j^t})_j$ of $(\varepsilon_k)_k$, possibly depending on t, with

$$(\pi_D, \boldsymbol{z}_{\varepsilon_{k_j^t}}(t), \nabla \boldsymbol{v}_{\varepsilon_{k_j^t}}(t))(P) \rightharpoonup (\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) \quad \text{2-weakly*},$$

as $j \to \infty$ and

$$\begin{aligned} &\limsup_{\varepsilon_k} [\langle \boldsymbol{\sigma}_{\varepsilon_k}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_k}(t)\rangle] = \\ &= \lim_{\varepsilon_{k_j^t}} [\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\rangle]; \end{aligned}$$

(c) $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is Θ-2-weakly* approximable from the left, for every $t \in \Theta$ (4.5.3) and (4.5.4) hold, and for every $t \in \Theta \setminus 0$ (4.3.3) and (4.3.4) hold for every ε_k.

In Theorem (4.6.8) we will prove that the evolution defined in this way satisfies the stationarity conditions (4), (5), and the energy inequality (3). Since the proof will be given using the language of Young measures, we translate the previous definition in terms of Young measures.

4.6.2. Approximable quasistatic evolution in terms of Young measures. The definition of approximable quasistatic evolution is now presented in terms of Young measures.

Definition 4.6.2. Given a boundary datum $\boldsymbol{\varphi} \in H^1([0,T]; H^1(D;\mathbb{R}^N))$, an external load $\boldsymbol{l} \in H^1([0,T]; H^1(D;\mathbb{R}^N)^*)$, an initial condition $(z_0, v_0) \in L^2(D;\mathbb{R}^m) \times \mathcal{A}(0)$ satisfying (4.3.1) and (4.3.2), and $T > 0$, an *approximable quasistatic evolution* of Young measures in the time interval $[0,T]$ is a pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$, for which there exist a positive sequence $\varepsilon_k \to 0$ and a subset Θ of $[0,T]$ with $0 \in \Theta$ and $\mathcal{L}^1([0,T]\setminus\Theta) = 0$, such that the solutions $(\boldsymbol{z}_{\varepsilon_k}, \boldsymbol{v}_{\varepsilon_k})$ of the ε_k-regularized problems satisfy the following conditions:

(a) for every finite sequence $t_1 < \cdots < t_n$ in Θ we have

$$\boldsymbol{\delta}_{(\boldsymbol{z}_{\varepsilon_k}(t_1), \ldots, \boldsymbol{z}_{\varepsilon_k}(t_n))} \rightharpoonup \boldsymbol{\mu}_{t_1 \ldots t_n} \quad \text{2-weakly*},$$

as $k \to \infty$;

(b) for every $t \in \Theta$, there exists a subsequence $(\varepsilon_{k_j^t})_j$ of $(\varepsilon_k)_k$, possibly depending on t, with

$$\boldsymbol{\delta}_{(\boldsymbol{z}_{\varepsilon_{k_j^t}}(t), \nabla \boldsymbol{v}_{\varepsilon_{k_j^t}}(t))} \rightharpoonup \boldsymbol{\nu}_t \quad \text{2-weakly*}, \tag{4.6.1}$$

as $j \to \infty$ and

$$\limsup_{\varepsilon_k}[\langle \boldsymbol{\sigma}_{\varepsilon_k}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_k}(t)\rangle] = \\ = \lim_{\varepsilon_{k_j^t}}[\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\rangle]; \tag{4.6.2}$$

(c) $\boldsymbol{\nu}$ is Θ-2-weakly*-approximable from the left, for every $t \in \Theta$ (4.5.3) and (4.5.4) hold, and for every $t \in \Theta \setminus 0$ (4.3.3) and (4.3.4) hold for every ε_k.

In the next subsection we will show that an evolution defined in this way, besides fulfiling the selection criterion mentioned in the introduction, satisfies also the stationarity condition and the energy inequality. In particular the technical condition (4.6.2) will be crucial to apply the argument in [**13**, Section 7].

Before stating this result, we clarify in which sense the notions of evolution given in terms of stochastic processes and in terms of Young measures are equivalent, and we make some technical remarks which will be useful in the proof of the main theorem.

Remark 4.6.3. As in Section 3.4.3, we can deduce that an approximable quasistatic evolution of Young measures belongs to $AY([0,T], \boldsymbol{\varphi})$; analogously, an approximable quasistatic evolution of stochastic processes is in $AY_{sp}([0,T], \boldsymbol{\varphi})$. In particular, if $(\boldsymbol{\nu}, \boldsymbol{\mu})$ is an approximable quasistatic evolution, then $\pi_{D\times\mathbb{R}^m}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$, for every $t \in [0,T]$.

Remark 4.6.4. If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes, the pair $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ defined by

$$\boldsymbol{\nu}_t := (\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) \quad \text{for every } t \in [0,T]$$
$$\boldsymbol{\mu}_{t_1\dots t_n} := (\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P) \quad \text{for every finite sequence } t_1 < \dots < t_n \text{ in } [0,T],$$

is an approximable quasistatic evolution of Young measures.

On the other side, thanks to Remark 4.6.3 and Theorem 2.2.4, given an approximable quasistatic evolution of Young measures $(\boldsymbol{\nu}, \boldsymbol{\mu})$ there exists a stochastic process $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ such that

$$(\pi_D, \boldsymbol{Z}_t, \boldsymbol{Y}_t)(P) = \boldsymbol{\nu}_t \quad \text{for every } t \in [0,T]$$
$$(\pi_D, \boldsymbol{Z}_{t_1}, \dots, \boldsymbol{Z}_{t_n})(P) = \boldsymbol{\mu}_{t_1\dots t_n} \quad \text{for every finite sequence } t_1 < \dots t_n \text{ in } [0,T];$$

in particular $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes.

Remark 4.6.5. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ is an approximable quasistatic evolution of Young measures, for every $t \in [0,T]$ there exists a unique function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that $\nabla\boldsymbol{v}(t) = \mathrm{bar}(\tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$, where $\mathrm{bar}(\tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t))$ denotes the barycentre of the Young measure $\tilde{\pi}_{\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t)$. This follows from Remark 4.6.3 and Remark 3.3.8.

Translating the previous remark in terms of stochastic processes we obtain the following result.

Remark 4.6.6. If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes, for every $t \in [0,T]$ there exists a unique function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that $\nabla\boldsymbol{v}(t) = \mathrm{bar}((\pi_D, \boldsymbol{Y}_t)(P))$.

REMARK 4.6.7. If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ is an approximable quasistatic evolution of Young measures, for every $t \in [0,T]$ we define

$$\boldsymbol{\sigma}(t,x) := \int_{Rm\times\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta, F)\; d\boldsymbol{\nu}_t^x(\theta, F), \tag{4.6.3}$$

$$\boldsymbol{\zeta}(t,x) := \int_{\mathbb{R}^m\times\mathbb{R}^{N\times d}} -\frac{\partial W}{\partial \theta}(\theta, F)\; d\boldsymbol{\nu}_t^x(\theta, F), \tag{4.6.4}$$

for a.e. $x \in D$. For every $t \in [0,T]$ we have that $\boldsymbol{\sigma}(t) \in L^2(D; \mathbb{R}^{N\times d})$ and $\boldsymbol{\zeta}(t) \in L^2(D; \mathbb{R}^m)$: this comes immediately from **(W.3)**, from $\pi_D(\boldsymbol{\nu}_t) = \mathcal{L}^d$, and from the fact that $\boldsymbol{\nu}_t \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})$. In the language of stochastic processes $\boldsymbol{\sigma}(t)$ and $\boldsymbol{\zeta}(t)$ can be characterized as the unique elements of $L^2(D; \mathbb{R}^{N\times d})$ and $L^2(D; \mathbb{R}^m)$, respectively, such that

$$\int_D \boldsymbol{\sigma}(t,x) g(x)\; dx = \int_{D\times\Omega} \frac{\partial W}{\partial F}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) g(x)\; dP(x,\omega), \tag{4.6.5}$$

$$\int_D \boldsymbol{\zeta}(t,x) h(x)\; dx = \int_{D\times\Omega} -\frac{\partial W}{\partial \theta}\big(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega)\big) h(x)\; dP(x,\omega), \tag{4.6.6}$$

for every $g \in L^2(D; \mathbb{R}^{N\times d})$, $h \in L^2(D; \mathbb{R}^m)$, where $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is the stochastic process corresponding to $(\boldsymbol{\nu}, \boldsymbol{\mu})$.

4.6.3. Properties of an approximable quasistatic evolution. The next theorem shows that an approximable quasistatic evolution of stochastic processes satisfies suitable properties of equilibrium, dual constraint, and energy inequality.

THEOREM 4.6.8. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, (z_0, v_0), ε_k, and $T > 0$ be as in the Definition (4.6.1). If $(\boldsymbol{Z}_t, \boldsymbol{Y}_t)_{t\in[0,T]}$ is an approximable quasistatic evolution of stochastic processes, then the following conditions are satisfied:*

- (ev0) *initial condition : $(\boldsymbol{Z}_0, \boldsymbol{Y}_0) = (z_0, v_0)$;*
- (ev1) *kinematic admissibility : for every $t \in [0,T]$, there exists a unique function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that $\nabla \boldsymbol{v}(t) = \mathrm{bar}((\pi_D, \boldsymbol{Y}_t)(P))$;*
- (ev2) *equilibrium condition : for every $t \in [0,T]$ and every $\tilde{u} \in H^1_{\Gamma_0}(0)$,*

$$\langle \boldsymbol{\sigma}(t), \nabla \tilde{u} \rangle = \langle \boldsymbol{l}(t), \tilde{u} \rangle;$$

- (ev3) *dual constraint : $\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0)$, for every $t \in [0,T]$;*
- (ev4) *energy inequality: for every $t \in [0,T]$ the map*

$$t \mapsto [\langle \boldsymbol{\sigma}(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t) \rangle],$$

where $\boldsymbol{v}(t)$ is the function appearing in (ev1)*, is integrable on $[0,T]$, and we have*

$$\int_{D\times\Omega} W(\boldsymbol{Z}_t(x,\omega), \boldsymbol{Y}_t(x,\omega))\; dP(x,\omega) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t) \rangle + \mathrm{Var}_H(\boldsymbol{Z}, P; 0, t) \le$$

$$\le \mathcal{W}(z_0, v_0) - \langle \boldsymbol{l}(0), v_0 \rangle + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle\; ds +$$

$$- \int_0^t [\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s) \rangle + \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}(s) \rangle]\; ds,$$

where $\mathrm{Var}_H(\boldsymbol{Z}, P; 0, t)$ is defined as in (3.3.22).

Thanks to Remark 4.6.4, to prove the previous theorem it is enough to prove the equivalent version for Young measures.

THEOREM 4.6.9. *Let $\boldsymbol{\varphi}$, $\boldsymbol{l}$, (z_0, v_0), ε_k, and $T > 0$ be as in the Definition (4.6.2). If $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]} \times SY^2_-([0,T], D; \mathbb{R}^m)$ is an approximable quasistatic evolution of Young measures, then the following conditions are satisfied:*

(ev0) *initial condition : $\boldsymbol{\nu}_0 = \boldsymbol{\delta}_{(z_0,v_0)}$;*

(ev1) *kinematic admissibility : for every $t \in [0,T]$, there exists a unique function $\boldsymbol{v}(t) \in \mathcal{A}(t)$ such that*

$$\nabla \boldsymbol{v}(t) = \mathrm{bar}(\pi_{D\times\mathbb{R}^{N\times d}}(\boldsymbol{\nu}_t)); \tag{4.6.7}$$

(ev2) *equilibrium condition : for every $t \in [0,T]$ and every $\tilde{u} \in H^1_{\Gamma_0}(0)$,*

$$\langle \boldsymbol{\sigma}(t), \nabla\tilde{u}\rangle = \langle \boldsymbol{l}(t), \tilde{u}\rangle; \tag{4.6.8}$$

(ev3) *dual constraint : $\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0)$, for every $t \in [0,T]$;*

(ev4) *energy inequality: for every $t \in [0,T]$ the map*

$$t \mapsto [\langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t)\rangle], \tag{4.6.9}$$

where $\boldsymbol{v}(t)$ is the function appearing in (ev1)*, is integrable on* $[0,T]$*, and we have*

$$\begin{aligned}\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta, F)\, d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu}; 0, t) \le \\ \le \mathcal{W}(z_0, v_0) - \langle \boldsymbol{l}(0), v_0\rangle + \int_0^t \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\, ds + \\ - \int_0^t [\langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle + \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}(s)\rangle]\, ds,\end{aligned}$$

where $\mathrm{Var}_H(\boldsymbol{\mu}; 0, t)$ *is defined as in (2.1.8).*

PROOF. Let $(\boldsymbol{\nu}, \boldsymbol{\mu})$ be an approximable quasistatic evolution of Young measures.
Condition (ev0) follows immediately from condition (b) of Definition (4.6.2) and $(\mathrm{ev0})_{\varepsilon_{k^0_j}}$.
Condition (ev1) has been proved in Remark 4.6.5.
We now prove (ev2); we observe that condition (b) and **(W.3)** imply that

$$\boldsymbol{\sigma}_{\varepsilon_{k^t_j}}(t) \rightharpoonup \boldsymbol{\sigma}(t) \quad \text{weakly in } L^2(D; \mathbb{R}^{N\times d}), \tag{4.6.10}$$

for every $t \in \Theta$, where $(\varepsilon_{k^t_j})_j$ is the sequence appearing in (b). Hence, for every $t \in \Theta \setminus 0$ (4.6.8) follows from $(\mathrm{ev2})_{\varepsilon_{k^t_j}}$, (4.6.10), and condition (c) of Definition 4.6.2, while for $t = 0$ is a direct consequence of (4.3.1), $(\mathrm{ev0})_{\varepsilon_{k^0_j}}$, and (4.6.10). If $t \in [0,T] \setminus \Theta$, let $s^j \le t$ be a sequence satisfying (2.1.2); from (4.6.8) for s^j, we can obtain (4.6.10) for t, using the continuity of the map $\boldsymbol{l}\colon [0,T] \to H^1(D; \mathbb{R}^N)^*$.

We show now (ev3). As for $\boldsymbol{\sigma}$ it is easy to see that

$$\boldsymbol{\zeta}_{\varepsilon_{k^t_j}}(t) \rightharpoonup \boldsymbol{\zeta}(t) \quad \text{weakly in } L^2(D; \mathbb{R}^m), \tag{4.6.11}$$

for every $t \in \Theta$, where $(\varepsilon_{k_j^t})_j$ is the sequence in (b). Thanks to (c), (4.6.11) implies that $\boldsymbol{\zeta}_{\varepsilon_{k_j^t}}(t) - \varepsilon_{k_j^t}\dot{\boldsymbol{z}}_{\varepsilon_{k_j^t}}(t) \rightharpoonup \boldsymbol{\zeta}(t)$ weakly in $L^2(D;\mathbb{R}^m)$, for every $t \in \Theta$, and thus, since $\partial\mathcal{H}(0)$ is sequentially weakly closed in $L^2(D;\mathbb{R}^m)$, we obtain from (c) and $(\mathrm{ev3})_{\varepsilon_{k_j^t}}$ that

$$\boldsymbol{\zeta}(t) \in \partial\mathcal{H}(0), \tag{4.6.12}$$

for every $t \in \Theta \setminus 0$, while for $t = 0$ it comes immediately from (4.3.2), $(\mathrm{ev0})_{\varepsilon_{k_j^0}}$, and(4.6.11). For $t \in [0,T] \setminus \Theta$, (4.6.12) follows now easily from (c).

Finally we want to prove (ev4). First of all we observe that if $(\varepsilon_{k_j^t})_j$ is the sequence appearing in (b), we have

$$\boldsymbol{v}_{\varepsilon_{k_j^t}}(t) \rightharpoonup \boldsymbol{v}(t) \quad \text{weakly in } H^1(D;\mathbb{R}^N); \tag{4.6.13}$$

hence

$$\begin{aligned} &\langle \boldsymbol{\sigma}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}(t)\rangle = \\ &= \limsup_k [\langle \boldsymbol{\sigma}_{\varepsilon_k}(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_{\varepsilon_k}(t)\rangle] \end{aligned} \tag{4.6.14}$$

for every $t \in \Theta$, thanks to (4.6.10), (4.6.13), and (4.6.2). Therefore the map (4.6.9) is measurable on $[0,T]$. Moreover, from Lemma 4.5.2 we deduce that

$$\begin{aligned} &|\langle \boldsymbol{\sigma}_\varepsilon(t), \nabla\dot{\boldsymbol{\varphi}}(t)\rangle - \langle \dot{\boldsymbol{l}}(t), \boldsymbol{v}_\varepsilon(t)\rangle| \le \\ &\le C'[\|\nabla\dot{\boldsymbol{\varphi}}(t)\|_2 + \|\dot{\boldsymbol{l}}(t)\|_{(H^1)^*}]; \end{aligned} \tag{4.6.15}$$

hence, thanks to the hypotheses on $\boldsymbol{\varphi}$ and $\boldsymbol{l}$ and to (4.6.14), the map (4.6.9) is integrable on $[0,T]$.

Fix $t \in \Theta$ and let $(\varepsilon_{k_j^t})_j$ be the sequence appearing in (b); since the term containing W is weakly lower semicontinuous and the variation is weakly lower semicontinuous too, thanks to condition (a) of Definition 4.6.2, we have

$$\begin{aligned} &\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\, d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \mathrm{Var}_H(\boldsymbol{\mu};0,t) \le \\ &\le \liminf_j \Big[\mathcal{W}(\boldsymbol{z}_{\varepsilon_{k_j^t}}(t), \boldsymbol{v}_{\varepsilon_{k_j^t}}(t)) - \langle \boldsymbol{l}(t), \boldsymbol{v}_{\varepsilon_{k_j^t}}(t)\rangle + \mathrm{Var}_H(\boldsymbol{z}_{\varepsilon_{k_j^t}};0,t)\Big]. \end{aligned}$$

Using $(\mathrm{ev4})_{\varepsilon_{k_j^t}}$ and (4.2.1), we deduce that

$$\begin{gathered} \int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta,F)\, d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \\ \mathrm{Var}_H(\boldsymbol{\mu};0,t) \le \\ \le \mathcal{W}(z_0, v_0) - \langle \boldsymbol{l}(0), v_0\rangle - \int_0^t \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle\, ds + \\ + \limsup_j \int_0^t [\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_{\varepsilon_{k_j^t}}(s)\rangle]\, ds. \end{gathered}$$

We can deduce, using Fatou Lemma thanks to (4.6.15), that

$$\limsup_j \int_0^t [\langle \boldsymbol{\sigma}_{\varepsilon_{k_j^t}}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_{\varepsilon_{k_j^t}}(s)\rangle] \, ds \leq$$
$$\leq \limsup_k \int_0^t [\langle \boldsymbol{\sigma}_{\varepsilon_k}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_{\varepsilon_k}(s)\rangle] \, ds \leq$$
$$\leq \int_0^t \limsup_k [\langle \boldsymbol{\sigma}_{\varepsilon_k}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}_{\varepsilon_k}(s)\rangle] \, ds.$$

Thanks to (4.6.2) this implies that

$$\begin{gathered}\int_{D\times\mathbb{R}^m\times\mathbb{R}^{N\times d}} W(\theta, F) \, d\boldsymbol{\nu}_t(x,\theta,F) - \langle \boldsymbol{l}(t), \boldsymbol{v}(t)\rangle + \\ \mathrm{Var}_H(\boldsymbol{\mu}; 0, t) \leq \\ \leq \mathcal{W}(z_0, v_0) - \langle \boldsymbol{l}(0), v_0\rangle - \int_0^t \langle \boldsymbol{l}(s), \dot{\boldsymbol{\varphi}}(s)\rangle \, ds + \\ + \int_0^t [\langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle - \langle \dot{\boldsymbol{l}}(s), \boldsymbol{v}(s)\rangle] \, ds.\end{gathered} \tag{4.6.16}$$

Let now $t \in [0,T]\setminus\Theta$ and let $s^j \to t$ be a sequence satisfying (2.1.2); it is easy to verify that

$$\mathrm{Var}_H(\boldsymbol{\mu}; 0, t) \leq \liminf_j \mathrm{Var}_H(\boldsymbol{\mu}; 0, s^j),$$

hence (ev2) for t can be deduced from (4.6.16) for s^j. $\square$

The following result is an existence theorem for approximable quasistatic evolution of stochastic processes.

THEOREM 4.6.10. *Given an external load $\boldsymbol{l} \in H^1([0,T]; H^1(D;\mathbb{R}^N)^*)$, a boundary datum $\boldsymbol{\varphi} \in H^1([0,T]; H^1(D;\mathbb{R}^N))$, an initial condition $(z_0, v_0) \in L^2(D;\mathbb{R}^m)\times\mathcal{A}(0)$ satisfying (4.3.1) and (4.3.2), and $T > 0$, there exists an approximable quasistatic evolution of stochastic processes (or of Young measures) in the time interval $[0,T]$.*

PROOF. Thanks to Remark (4.6.4), it is enough to prove that there exists an approximable quasistatic evolution of Young measures. Fixed a positive sequence $\varepsilon_k \to 0$, let $(z_{\varepsilon_k}, v_{\varepsilon_k})$ be the solution of the ε_k-regularized problem in the time interval $[0,T]$, with external load $\boldsymbol{l}$, boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0, v_0). Thanks to (4.4.15) and **(H.2)** we are in the hypothesis of Helly's Theorem for compatible systems of Young measures (see Theorem 2.1.6). Therefore, by passing to a subsequence still denoted by $(\varepsilon_k)_k$, we can conclude that there exist $\Theta \subset [0,T]$, with $0 \in \Theta$ and $\mathcal{L}^1([0,T]\setminus\Theta) = 0$, and $\boldsymbol{\mu} \in SY^2_-([0,T], D; \mathbb{R}^m)$, which satisfy condition (a) of Definition 4.6.2.

Thanks to Remark 4.5.3, we can assume that (4.5.3), (4.5.4) hold for every $t \in \Theta$, by choosing a subset of Θ if necessary; analogously we can assume that (4.3.3) and (4.3.4) hold for every $t \in \Theta\setminus 0$ and every ε_k. For every $t \in \Theta$ select a subsequence $(\varepsilon_{k_j^t})_j$ of $(\varepsilon_k)_k$ which satisfies (4.6.2); thanks to (4.4.15) and (4.4.16), we can apply Lemma 2.1.9 to the sequence of compatible systems $(\boldsymbol{\delta}_{(\boldsymbol{z}_{\varepsilon_k}, \nabla\boldsymbol{v}_{\varepsilon_k})})_k$ and we obtain a family of Young measures

$\boldsymbol{\nu} \in Y^2(D; \mathbb{R}^m \times \mathbb{R}^{N\times d})^{[0,T]}$, which is Θ-2-weakly* approximable from the left and satisfies (4.6.1), for a suitable subsequence of $(\varepsilon_{k_j^t})_j$. This proves (b) and (c).

$\square$

4.7. A finite dimensional example

In this section we will propose the complete analysis of the approximable quasistatic evolution for a concrete case, in which the hypotheses of Remark 4.5.1 are fulfiled and hence the internal variable and the gradient of the deformation are functions from $[0,T]$ into a finite dimensional space.

Let consider the case $d = N = m = 1$, $D = (0,1)$, and $\Gamma_0 = \{0,1\}$. We assume $H = |\cdot|$, $\boldsymbol{l} \equiv 0$, and

$$W(\theta, y) := \tfrac{1}{10}[\eta(y)(y - a(\theta))^2 + (1 - \eta(y))y^2] + b(\theta) \quad \text{for every } \theta, y \in \mathbb{R}, \tag{4.7.1}$$

where $a \in \mathcal{C}^2(\mathbb{R})$ is bounded with its first and second derivative and $a(\theta) = \theta$ if $|\theta| \le 2$, $\eta \in \mathcal{C}^2_c(\mathbb{R})$ is a cut off-function with $\eta(y) = y$ if $|y| \le 7 + 5\,b'(2)$, and b is a $\mathcal{C}^2$ function satisfying the following properties, for every $\theta \in \mathbb{R}$ (see Figure 1):

- (b.1) $b(\theta) \ge c\theta^2 + d$, for suitable positive constant c, d;
- (b.2) $b(\theta) + |\theta + 1| > 2$, for every $\theta \ne -1$;
- (b.3) b has a local minimum at -1, with $b(-1) = 2$, a global minimum at 1, with $b(1) < 1$, and a local maximum in 0, and there are no other local extrema;
- (b.4) $5\,b'' + 1$ is bounded and has exactly two zeros, $-1 < \theta_1 < 0 < \theta_2 < 1$, with $b'(\theta_1) < b'(2)$.

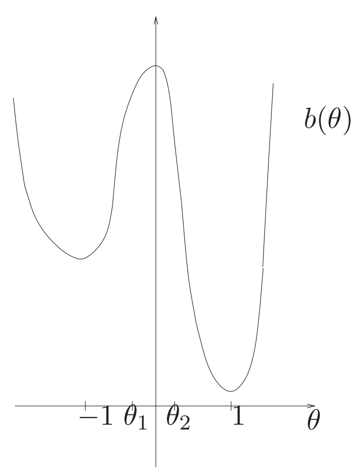

FIGURE 1. The function b

It is immediate to verify that such a W satisfies hypotheses **(W.1)** and **(W.3)**.

Let us fix T such that $6 + \theta_1 + 5\,b'(\theta_1) < T < 8 + 5\,b'(2)$; we will study the approximable quasistatic evolution in the time interval $[0,T]$ with $\boldsymbol{\varphi}(t,x) := (t-1)x$ for every $t \in [0,T]$ and every $x \in [0,1]$ (which corresponds to the boundary condition $\boldsymbol{v}(t,0) = 0$ and $\boldsymbol{v}(t,1) = t - 1$), and initial condition $(z_0, v_0) = (-1, \boldsymbol{\varphi}(0))$.

THEOREM 4.7.1. *Let W, $\boldsymbol{l}$, H, $\boldsymbol{\varphi}$, T, and (z_0, v_0) satisfy the assumptions at the beginning of this section. Then the unique approximable quasistatic evolution corresponding to this data is given by*

$$\boldsymbol{v}(t,x) = \boldsymbol{\varphi}(t,x) \quad \textit{for every } t \in [0,T];$$
$$\boldsymbol{z}(t,x) = z(t) := \begin{cases} -1 & \textit{for } 0 \le t \le 5 \\ z_1(t) & \textit{for } 5 < t \le t_1 \\ z_2(t) & \textit{for } t_1 < t \le T \end{cases} \tag{4.7.2}$$

where $t_1 := 6 + \theta_1 + 5\, b'(\theta_1)$, for every $t \in [5, t_1]$ $z_1(t)$ is the unique solution in the interval $[-1, \theta_1]$ of the equation

$$\tfrac{1}{5}(t - 1 - \theta(t)) - b'(\theta(t)) = 1, \tag{4.7.3}$$

and for $t \in (t_1, T]$ z_2 is the unique solution of (4.7.3).

PROOF. We are in the case of Remark 4.5.1, hence the solution $(\boldsymbol{v}_\varepsilon, \boldsymbol{z}_\varepsilon)$ of the ε-regularized problem is

$$\boldsymbol{v}_\varepsilon(t,x) := \boldsymbol{\varphi}(t,x), \tag{4.7.4}$$
$$\boldsymbol{z}_\varepsilon(t,x) = z_\varepsilon(t), \tag{4.7.5}$$

where z_ε is the solution of the Cauchy problem

$$\begin{cases} \dot z_\varepsilon(t) = \frac{1}{\varepsilon}\big[- W_\theta(\boldsymbol{z}_\varepsilon(t), t-1) - P_{[-1,1]}\big(- W_\theta(\boldsymbol{z}_\varepsilon(t), t-1)\big)\big] \\ z_\varepsilon(0) = -1, \end{cases} \tag{4.7.6}$$

where $P_{[-1,1]}$ is the projection on the interval $[-1,1]$. By the upper bound on T, $-W_\theta(\theta, t-1)$ takes the form

$$g(t,\theta) := \tfrac{1}{5}(t - 1 - \theta) - b'(\theta),$$

for every $t \in [0,T]$ and for every $|\theta| \le 2$. Hence the equation in (4.7.6) becomes

$$\varepsilon \dot z_\varepsilon(t) = \begin{cases} g(t, z_\varepsilon(t)) - 1 & \text{if} \quad g(t, z_\varepsilon(t)) > 1 \\ 0 & \text{if} \quad |g(t, z_\varepsilon(t))| \le 1 \\ g(t, z_\varepsilon(t)) + 1 & \text{if} \quad g(t, z_\varepsilon(t)) < -1 \end{cases} \tag{4.7.7}$$

until $|z_\varepsilon(t)| \le 2$. For every ε let t_ε be the greatest time in $[0,T]$ such that $|z_\varepsilon(t)| \le 2$ for every $t \in [0, t_\varepsilon]$. In particular $z_\varepsilon(t_\varepsilon) = 2$. Since $0 \le g(t,-1) = \frac{1}{5}t \le 1$ for $t \in [0,5]$, we have $z_\varepsilon(t) = -1$, for every $t \le 5$ and every ε. In particular we have $g(5, z_\varepsilon(5)) = 1$ and $t_\varepsilon > 5$. It is easy to see that $g(t, z_\varepsilon(t)) \ge 1$ for $t > 5$. Indeed let U_ε be the open set $\{t \in (5, t_\varepsilon) : g(t, z_\varepsilon(t)) < 1\}$ and let (α, β) be any connected component of U_ε. Since $g(\alpha, z_\varepsilon(\alpha)) = 1$, there exists $0 < \delta_\varepsilon < \beta - \alpha$ such that $0 < g(t, z_\varepsilon(t)) < 1$ for every $t \in (\alpha, \alpha + \delta_\varepsilon)$; therefore $\dot z_\varepsilon(t) = 0$ for every $t \in (\alpha, \alpha + \delta_\varepsilon)$, in particular $z_\varepsilon(t) = z_\varepsilon(\alpha)$. Since $g(\cdot, z_\varepsilon(\alpha))$ is strictly increasing (indeed $\frac{\partial g}{\partial t}(t,\theta) = \frac{1}{5}$), we have $1 = g(\alpha, z_\varepsilon(\alpha)) < g(t, z_\varepsilon(\alpha)) = g(t, z_\varepsilon(t))$ for every $t \in (\alpha, \alpha + \delta_\varepsilon)$, which contradicts $g(t, z_\varepsilon(t)) < 1$. Hence $U_\varepsilon = \emptyset$ and $g(t, z_\varepsilon(t)) \ge 1$ for every $t \in [5, t_\varepsilon]$. Thanks to the upper bound on T we have $g(T, 2) < 1$, but, if $t_\varepsilon < T$, we have $1 \le g(t_\varepsilon, z_\varepsilon(t_\varepsilon)) = g(t_\varepsilon, 2) < g(T, 2)$ which contradicts $g(T,2) < 1$. Therefore $t_\varepsilon = T$ for every ε, and $g(t, z_\varepsilon(t)) \ge 1$ for every $t \in [5, T]$.

Hence we can conclude that, for $t \in (5, T]$, z_ε is the unique solution of the equation

$$\varepsilon \dot z_\varepsilon(t) = g(t, z_\varepsilon(t)) - 1. \tag{4.7.8}$$

Note that

$$\frac{\partial g}{\partial \theta}(t,\theta) = -\tfrac{1}{5}(1 + 5\,b''(\theta)),$$

for every t and θ. Therefore from (b.4) we know that $\partial g/\partial\theta$ has exactly two zeros θ_1 and θ_2 with $-1 < \theta_1 < 0 < \theta_2 < 1$.

First of all we want to show that there exists a unique solution $z_1(t) \in (-1, \theta_1)$ to the equation

$$g(t, z(t)) = 1, \tag{4.7.9}$$

for $t \in (5, t_1)$ where $t_1 = 6 + \theta_1 + 5\,b'(\theta_1)$. Note that $g(t_1, \theta_1) = 1$; since $g(\cdot, \theta)$ is strictly increasing for every θ, we have $g(t, -1) > g(5, -1) = 1 = g(t_1, \theta_1) > g(t, \theta_1)$ for every $t \in (5, t_1)$; hence for every $t \in (5, t_1)$ there exists a unique $z_1(t) \in (-1, \theta)$ solving (4.7.9) (because $\frac{\partial g}{\partial \theta}(t, \cdot)$ never vanishes on $(-1, \theta_1)$). By the Inverse Function Theorem the map $t \mapsto z_1(t)$ is $\mathcal{C}^1$ and we can deduce that $\lim_{t\to t_1^-} z_1(t) = \theta_1$ (indeed if not, let θ^* be this limit; we have $-1 \le \theta^* < \theta_1$ and $g(t_1, \theta^*) = 1$, which contradicts the fact that $g(t_1, \cdot)$ is strictly decreasing on $(-1, \theta_1)$).

It is easy to see that for $t > t_1$ the equation (4.7.9) has a unique solution: indeed we can write

$$g(t,\theta) - 1 = \tfrac{1}{5}(t-1) - \psi(\theta),$$

where $\psi(\theta) := \frac{1}{5}\theta + b'(\theta) + 1$; since, for every $t > t_1$, $\psi(\theta) \le \psi(\theta_1) = \frac{1}{5}(t_1 - 1) < \frac{1}{5}(t-1)$, for every $\theta \le \theta_2$, and $\lim_{\theta\to+\infty} \psi(\theta) = +\infty$, we deduce that the zeros of $\frac{1}{5}(t-1) - \psi(\theta)$ exist and are contained in $(\theta_2, +\infty)$; since in this interval $\frac{\partial g}{\partial \theta}$ never vanishes, we can apply again the Inverse Function Theorem to obtain the existence of a unique continuous function $z_2 \colon (t_1, T] \to \mathbb{R}$ solving (4.7.9).

We want now to show that z_1 is the unique approximable quasistatic evolution in $[5, t_1]$. First of all we observe that, since $z_\varepsilon(5) = z_1(5) = -1$ and $\varepsilon \dot z_1(t) > 0 = g(t, z_1(t)) - 1$ while $\varepsilon \dot z_\varepsilon(t) = g(t, z_\varepsilon(t)) - 1$, by the comparison principle $z_1(t) \ge z_\varepsilon(t)$ for every $t \in [5, t_1)$.Let now fix $\eta > 0$ and $\bar t \in (5, t_1)$; if we show that there exists ε_0 such that for every $\varepsilon \le \varepsilon_0$ we have $z_\varepsilon(t) \in [z_1(t) - \eta, z_1(t)]$ for every $t \in (5, \bar t)$, we can conclude that $z(t) = z_1(t)$ on $(5, \bar t)$. Let

$$c_\eta := \min_{t\in[5,\bar t]} g(t, z_1(t) - \eta) - 1$$
$$m := \max_{t\in[5,\bar t]} \dot z_1(t);$$

we have $m < +\infty$ by continuity of $\dot z_1$, and $c_\eta > 0$ because $g(t, z_1(t) - \eta) > g(t, z_1(t)) = 1$. Therefore we can find $\varepsilon_0 > 0$ such that $\varepsilon_0\, m < c_\eta$, and for every $\varepsilon \le \varepsilon_0$ we have $z_1(5) - \eta < z_\varepsilon(5)$, $\varepsilon \dot z_1(t) < g(t, z_1(t) - \eta) - 1$ while $\varepsilon \dot z_\varepsilon(t) = g(t, z_\varepsilon(t)) - 1$, hence $z_\varepsilon(t) \ge z_1(t) - \eta$ for every $t \in (5, \bar t)$.

Since z is left continuous by definition we can conclude that $z(t) = z_1(t)$ for every $t \in [5, t_1]$.

Finally we show that z must coincide with z_2 on $(t_1, T]$.

As $|z_\varepsilon(t)| \le 2$ for every $t \in [0, T]$ and every ε, condition (ev3) satisfied by z can be written as

$$g(t, z(t)) \in [-1, 1]. \tag{4.7.10}$$

Since we have proved that $g(t, z_\varepsilon(t)) \geq 1$, for every $t > t_1$, it follows that z satisfies (4.7.9), for $t > t_1$. As this equation has a unique solution z_2 defined on (t_1, T), we can conclude that $z(t) = z_2(t)$ for every $t \in (t_1, T]$. $\square$

We prove now that the approximable quasistatic evolution described in Theorem 4.7.1 does not fulfil the requirements of the definition of globally stable quasistatic evolution given in Definition 3.3.12.

THEOREM 4.7.2. *The datum $(-1, \boldsymbol{\varphi}(0))$ is stable for the considered problem, but the approximable quasistatic evolution described in Theorem 4.7.1 does not satisfies partial-global stability, i.e. there exists $t \in [0, T]$, $\tilde{z} \in L^2((0,1))$, and $\tilde{u} \in H^1_0(0,1)$ with*

$$W(z(t), t-1) > \int_0^1 W(z(t) + \tilde{z}(x), t - 1 + \tilde{u}'(x))\, dx + \|\tilde{z}\|_1. \tag{4.7.11}$$

PROOF. First of all we have to verify that the initial condition satisfies the minimality condition requested in Definition 3.3.12. To this aim we have to check that

$$W(-1,-1) \leq \int_0^1 W(-1 + \tilde{z}(x), -1 + \tilde{u}'(x))\, dx + \|\tilde{z}\|_1,$$

for every $\tilde{z} \in L^2((0,1))$ and $\tilde{u} \in H^1_0(0,1)$; this is immediate because $W(-1,-1) = 2$, while

$$\begin{aligned} &\int_0^1 W(-1 + \tilde{z}(x), -1 + \tilde{u}'(x))\, dx + \|\tilde{z}\|_1 \geq \\ &\geq \int_0^1 [b(-1 + \tilde{z}(x)) + |\tilde{z}(x)|]\, dx \geq 2, \end{aligned}$$

thanks to assumption (b.2).

Let now consider $t \in (4,5]$, $\tilde{z} = 2$, and $\tilde{u} = 0$. We have $W(z(t), t-1) = \frac{1}{10}(t^2 + 20)$, while thanks to (b.3)

$$\begin{aligned} &\int_0^1 W(z(t) + \tilde{z}(x), t - 1 + \tilde{u}'(x))\, dx + \|\tilde{z}\|_1 = \\ &= W(1, t-1) + 2 = \tfrac{1}{10}(t-2)^2 + b(1) + 2 < \tfrac{1}{10}(t-2)^2 + 3 = \\ &\tfrac{1}{10}(t^2 - 4t + 34) < \tfrac{1}{10}(t^2 + 18). \end{aligned}$$

$\square$

CHAPTER 5

Globally stable quasistatic evolution in the discrete case

5.1. Introduction

In this chapter we analyze the particular case in which Z is a finite set $\{\theta_\alpha : \alpha = 1, \dots, q\}$, representing the different phases (or phase variants) of the crystal, and the internal variable $z\colon D \to Z$ represents the phase distribution of the material.

As before, v denotes the deformation, the stored energy of the system can be written as

$$\mathcal{W}(z, v) := \int_D W(z(x), \nabla v(x))\; dx,$$

while the energy dissipation associated to a phase transition is represented by

$$\int_D H(z_{new}(x), z_{old}(x))\; dx,$$

where H is a distance on Z, z_{old} is the old phase distribution and z_{new} the new one. We require that the admissible deformations satisfy a prescribed time-dependent boundary condition $\boldsymbol{\varphi}(t)$, which we impose on the whole boundary ∂D to avoid some technical difficulties; for the same reason, we neglect any contribution due to external forces.

Also in this case, the lack of convexity of the energy functional gives rise to many technical difficulties, making unsolvable in usual functional spaces the incremental minimum problems. We follow the same approach of Chapter 3, and set the problem in a suitable space of Young measures, where the incremental minimum problems can be solved.

The discrete setting allows to describe more explicitly Young measures and compatible systems, as explained in Section 2.3. The aim of the present chapter is to prove an existence result for the quasistatic evolution in a time interval $[0, T]$, defined as a pair $(\boldsymbol{b}, \boldsymbol{\lambda}) = (\boldsymbol{b}, (\boldsymbol{\lambda}_t)_{t\in[0,T]})$ satisfying an admissibility condition and suitably reformulated stability condition and energy inequality. The notion of evolution considered here seems to be better than the one proposed in Chapter 3: the partial-global stability is a minimality condition with respect to a quite large class of competitors and the energy inequality involves any pair of time instants.

The proof of the existence theorem (Theorem 5.4.2) follows the classical scheme of time-discretization, resolution of incremental minimum problems, and passage to the limit of the approximate solutions.

The new feature concerns the choice of the solutions to the discretized minimum problems: the regularity results for quasi-minima of integral functional are used to prove a uniform bound over the moments of order $2r > 2$ of the selected minimizers, and consequently of the approximate solutions $(\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t)$. As a by-product of this selection, we get the continuity of the functional

$$(\boldsymbol{b}_n^t, \boldsymbol{\lambda}_n^t) \mapsto \int_{D\times\mathbb{R}^{N\times d}} \sum_\alpha (\boldsymbol{b}_n^t)_\alpha(x) W(\theta_\alpha, F)\, d(\boldsymbol{\lambda}_n^t)_\alpha(x,F),$$

which is crucial in order to prove the existence result.

5.2. The mechanical model

The *reference configuration* D satisfies all the assumptions in Section 3.2, except for condition (3.2.1), which is not necessary here.

We indicate the deformation with v and the internal variable with z. We denote the stored energy density by $W\colon Z\times\mathbb{R}^{N\times d} \to [0,+\infty)$, while the dissipation rate density will be a metric on Z^2, denoted by $H\colon Z^2 \to [0,+\infty)$. The energy density W satisfies the assumption **(W.1)** of Section 3.2, while **(W.2)** is substituted by the following weaker condition:

(W.4) $W(\theta,\cdot)$ is of class $\mathcal{C}^2$ and

$$\Big|\frac{\partial W}{\partial F}(\theta,F)\Big| \le C(1+|F|).$$

The integral functional associated to W and H are denoted by $\mathcal{W}$ and $\mathcal{H}$, respectively.

Given two distinct times $s<t$, the *global dissipation* of a function $\boldsymbol{z}\colon [0,T]\to L^\infty(D;Z)$ in the interval $[s,t]$ will be

$$\mathrm{Var}_H(\boldsymbol{z};s,t) := \sup \sum_{i=1}^k \mathcal{H}(\boldsymbol{z}(\tau_i), \boldsymbol{z}(\tau_{i-1})),$$

where the supremum will be taken among all finite partitions $s=\tau_0<\tau_1<\cdots<\tau_k=t$.

The prescribed boundary datum on ∂D at time t is denoted by $\boldsymbol{\varphi}(t)$; we assume $\boldsymbol{\varphi}\in AC([0,T];W^{1,p}(D;\mathbb{R}^N))$, with $2<p<+\infty$.

The kinematically admissible values for the deformation at time t are contained in $\mathcal{A}(t)$, where $\mathcal{A}(t) := \boldsymbol{\varphi}(t) + H_0^1(D;\mathbb{R}^N)$.

5.3. Admissible set in terms of Young measures

Definition 5.3.1. Given $A\subset\mathbb{R}$ and $\boldsymbol{w}\colon A\to H^1(D;\mathbb{R}^N)$, we define the admissible set for the time set A and the boundary datum $\boldsymbol{w}$, $Ad(A,q,\boldsymbol{w})$, as the set of all pairs $(\boldsymbol{b},\boldsymbol{\lambda})\in S(A,D,q)\times(Y(D;\mathbb{R}^{N\times d})^q)^A$ such that property (2.3.10) (for $p=2$) is satisfied by $\boldsymbol{b}_\alpha^t\boldsymbol{\lambda}_\alpha^t$, for every α and t, and the following condition holds: for every finite sequence $t_1<\cdots<t_n$ in A, for every $i=1,\dots,n$, and every $k\in\mathbb{N}$, there exist a measurable partition $(D_\alpha^{i,k})_{\alpha=1}^q$ of D and a function $v_i^k\in\boldsymbol{w}(t_i)+H_0^1(D;\mathbb{R}^N)$ such that:

(1) for every $(\alpha_1,\dots,\alpha_n)\in\mathscr{A}_n^q$

$$1_{D_{\alpha_1}^{1,k}}\cdots 1_{D_{\alpha_n}^{n,k}} \rightharpoonup \boldsymbol{b}_{\alpha_1\dots\alpha_n}^{t_1\dots t_n} \quad L^\infty\text{-weakly*, as } k\to\infty;$$

(2) for every $i=1,\dots,n$ there exists a subsequence $(k_j^i)_j$, possibly depending on i, such that

$$1_{D_\alpha^{i,k_j^i}}\boldsymbol{\delta}_{\nabla v^i_{k_j^i}} \rightharpoonup \boldsymbol{b}_\alpha^{t_i}\boldsymbol{\lambda}_\alpha^{t_i} \quad 2\text{-weakly*, as } j\to\infty$$

for every $\alpha=1,\dots,q$.

The following remark compares the notion of $Ad(A, q, \boldsymbol{w})$ with the notion of admissible set in terms of Young measures $AY(A, Z, \boldsymbol{w})$, as defined in Section 3.3.2.

REMARK 5.3.2. Given $A \subset \mathbb{R}$ and $\boldsymbol{w}\colon A \to H^1(D;\mathbb{R}^N)$, let us consider $(\boldsymbol{b},\boldsymbol{\lambda}) \in S(A,D,q) \times (Y(D;\mathbb{R}^{N\times d})^q)^A$, with $(\boldsymbol{b}^t,\boldsymbol{\lambda}^t)$ satisfying (2.3.10) for $p=2$, and $(\boldsymbol{\nu},\boldsymbol{\mu}) \in Y^2(D;Z\times\mathbb{R}^{N\times d})^A \times SY(A,D;Z)$ defined by

$$\boldsymbol{\nu}_{t_i} = \sum_{\alpha=1}^{q} \boldsymbol{b}^{t_i}_{\alpha}(\delta_{\theta_\alpha} \otimes \boldsymbol{\lambda}^{t_i}_{\alpha}), \quad \text{for every } t \in A$$
$$\boldsymbol{\mu}_{t_1\dots t_n} = \sum_{(\alpha_1,\dots,\alpha_n)} \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n} \boldsymbol{\delta}_{(\theta_{\alpha_1},\dots,\theta_{\alpha_n})} \quad \text{for every } t_1 < \cdots < t_n \text{ in } A.$$

Then $(\boldsymbol{b},\boldsymbol{\lambda}) \in Ad(A,q,\boldsymbol{w})$ if and only if $(\boldsymbol{\nu},\boldsymbol{\mu}) \in AY(A,Z,\boldsymbol{w})$, i.e. for every finite sequence $t_1 < \cdots < t_n$ in A there exist sequences $(z^k_i)_k \in L^\infty(D;Z)$, $(v^k_i)_k \subset \boldsymbol{w}(t_i) + H^1_0(D;\mathbb{R}^N)$, for $i=1,\dots,n$ such that

$(\mathrm{app1})_Z$ we have
$$\boldsymbol{\delta}_{(z^k_1,\dots,z^k_n)} \rightharpoonup \boldsymbol{\mu}_{t_1\dots t_n} \quad \text{weakly*}, \tag{5.3.1}$$
as $k \to \infty$;

$(\mathrm{app2})_Z$ for every $i=1,\dots,n$, there exists a sequence of integers $(k^i_j)_j$, possibly depending on i, such that
$$\boldsymbol{\delta}_{(z_i^{k^i_j},\nabla v_i^{k^i_j})} \rightharpoonup \boldsymbol{\nu}_{t_i} \quad \text{2-weakly*}, \tag{5.3.2}$$
as $j \to \infty$.

Indeed, given $(D^{i,k}_\alpha)_\alpha$ satisfying the approximation property for $\boldsymbol{b}^{t_i}_\alpha$, we can define z^k_i by $z^k_i(x) = \theta_\alpha$ whenever $x \in D^{i,k}_\alpha$, or equivalently, given z^k_i satisfying the approximation property for ν_{t_i}, we can consider $D^{i,k}_\alpha := \{x \in D \ : \ z^k_i(x) = \theta_\alpha\}$, for $\alpha = 1,\dots,q$.

The closure properties of $Ad(A,q,\boldsymbol{w})$ are described by the following lemma, which is the formulation in our discrete setting of Lemma 3.3.7.

LEMMA 5.3.3. *Let $(\boldsymbol{w}^j)_j$ be a sequence of functions from A into $H^1(D,\mathbb{R}^m)$, such that $\boldsymbol{w}^j(t) \to \boldsymbol{w}(t)$ strongly in H^1, for every $t \in A$ and let $(\boldsymbol{b},\boldsymbol{\lambda}) \in S(A,D,q)\times(Y(D;\mathbb{R}^{N\times d})^q)^A$ with $(\boldsymbol{b}^t,\boldsymbol{\lambda}^t)$ satisfying (2.3.10) for $p=2$, for every $t \in A$. Assume that for every finite sequence $t_1 < \cdots < t_n$ in A there exists a sequence $(\boldsymbol{b}^j,\boldsymbol{\lambda}^j) \in Ad(\{t_1,\dots,t_n\},q,\boldsymbol{w}^j)$ such that*
$$(\boldsymbol{b}^j)^{t_1\dots t_n}_{\alpha_1\dots\alpha_n} \rightharpoonup \boldsymbol{b}^{t_1\dots t_n}_{\alpha_1\dots\alpha_n} \quad L^\infty\text{-weakly } *, \tag{5.3.3}$$
as $j \to \infty$ for every $(\alpha_1,\dots,\alpha_n) \in \mathscr{A}^q_n$, and such that for every i there exists a sequence of integers $(j^i_h)_h$, possibly depending on i, satisfying
$$((\boldsymbol{b}^{j^i_h})^{t_i}_\alpha(\boldsymbol{\lambda}^{j^i_h})^{t_i}_\alpha) \rightharpoonup \boldsymbol{b}^{t_i}_\alpha\boldsymbol{\lambda}^{t_i}_\alpha, \quad \text{2-weakly*}, \tag{5.3.4}$$
as $h \to \infty$ for every $\alpha = 1,\dots,q$. Then $(\boldsymbol{b},\boldsymbol{\lambda}) \in Ad(A,q,\boldsymbol{w})$.

The following lemma will be used in order to provide a class of competitors for the discretized minimum problem in Section 5.5.1.

Lemma 5.3.4. *Let* $0 \le t_1 < \cdots < t_m \le T$ *be a finite sequence in* A. *For every* $j = 1, \dots, m$, *consider* $v_j \in \boldsymbol{w}(t_j) + H^1_0(D; \mathbb{R}^N)$ *and a measurable partition* $(D^j_\alpha)^q_{\alpha=1}$ *of* D. *Let* $M\colon D \to \mathbb{M}^{q\times q}_{St}$, $x \mapsto (M_{\beta\alpha}(x))_{\beta\alpha}$ *be a measurable map, and* $\tilde u$ *an element of* $H^1_0(D;\mathbb{R}^N)$.

Let $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in Y^2(D; Z \times \mathbb{R}^{N\times d})^{\{t_1,\dots,t_m\}} \times SY^2(\{t_1, \dots, t_m\}, D; Z)$ *be defined by*

$$\boldsymbol{\nu}_{t_m} := \sum_{\alpha,\beta} M_{\beta\alpha} 1_{D^m_\alpha} \boldsymbol{\delta}_{(\theta_\beta, \nabla v_m + \nabla \tilde u)},$$

$$\boldsymbol{\nu}_{t_j} := \sum_{\alpha=1}^{q} 1_{D^j_\alpha} \boldsymbol{\delta}_{(\theta_\alpha, \nabla v_j)} \quad \textit{for every } j < m,$$

$$\boldsymbol{\mu}_{t_1 \dots t_m} := \sum_{\alpha_1,\dots,\alpha_{m-1},\alpha,\beta} M_{\beta\alpha} 1_{D^1_{\alpha_1}} \cdot \dots \cdot 1_{D^{m-1}_{\alpha_{m-1}}} \cdot 1_{D^m_\alpha} \boldsymbol{\delta}_{(\theta_{\alpha_1}, \dots, \theta_{\alpha_{m-1}}, \theta_\beta)}.$$

Then $(\boldsymbol{\nu}, \boldsymbol{\mu}) \in AY(\{t_1, \dots, t_m\}, Z, \boldsymbol{w})$.

Proof. Let us consider first the particular case of $M\colon D \to \mathbf{M}^{q\times q}_{St}$ constant.

Fix $\alpha \in \{1, \dots, q\}$ and define a measure $\boldsymbol{\nu}^\alpha_{t_m}$ on $D \times Z \times \mathbb{R}^{N\times d}$ by

$$\boldsymbol{\nu}^\alpha_{t_m} := \sum_{\beta} M_{\beta\alpha} 1_{D^m_\alpha} \boldsymbol{\delta}_{(\theta_\beta, \nabla \tilde v_m)},$$

where $\tilde v_m := v_m + \tilde u$. Consider a measurable partition $(S^\alpha_\beta)_\beta$ of the unitary cube $[0,1]^d$, with $|S^\alpha_\beta| = M_{\beta\alpha}$ for every β (it is possible to find such a partition since $0 \le M_{\beta\alpha} \le 1$ and $\sum_\beta M_{\beta\alpha} = 1$, by the hypotheses on M). Let us define now a measurable function $\tilde z^\alpha \colon [0,1]^d \to Z$ by setting

$$\tilde z^\alpha(x) = \theta_\beta \quad \text{for a.e. } x \in S^\alpha_\beta,$$

for every $\beta = 1, \dots, q$, and extend it by periodicity to all $\mathbb{R}^d$. For every $\delta > 0$, the function $\tilde z^\alpha_\delta \colon \mathbb{R}^d \to Z$ defined by $\tilde z^\alpha_\delta(x) := \tilde z^\alpha(\frac{x}{\delta})$, for a.e. $x \in \mathbb{R}^d$, is δ-periodic. By Lemma 1.1.2, we have

$$1_{\{x\in\mathbb{R}^d \,:\, \tilde z^\alpha_\delta(x) = \theta_\beta\}} \rightharpoonup M_{\beta\alpha} \quad L^\infty\text{-weakly*},$$

as $\delta \to 0$. Let now $\psi \in \mathcal{C}_0(D \times Z \times \mathbb{R}^{N\times d}) = \mathcal{C}_0(D \times \mathbb{R}^{N\times d})^Z$; we have

$$1_{D^m_\alpha}(x) \psi(x, \tilde z^\alpha_\delta(x), \nabla \tilde v_m(x)) = 1_{D^m_\alpha}(x) \sum_{\beta=1}^{q} \psi(x, \theta_\beta, \nabla \tilde v_m(x)) 1_{\{x\in\mathbb{R}^d \,:\tilde z^\alpha_\delta(x)=\theta_\beta\}}(x),$$

for a.e. $x \in D$, and the function $x \mapsto 1_{D^m_\alpha}(x)\psi(x, \theta_\beta, \nabla\tilde{v}_m(x))$ is in $L^1(D)$ for every β. Hence we can deduce that

$$\int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\ d(1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde{z}^\alpha_\delta,\nabla\tilde{v}_m)})(x,\theta,F) =$$

$$= \int_D 1_{D^m_\alpha}(x)\psi(x,\tilde{z}^\alpha_\delta(x),\nabla\tilde{v}_m(x))\ dx =$$

$$= \sum_\beta \int_{\mathbb{R}^d} 1_{D^m_\alpha}(x)\psi(x,\theta_\beta,\nabla\tilde{v}_m(x))1_{\{x\in\mathbb{R}^d\ :\tilde{z}^\alpha_\delta(x)=\theta_\beta\}}(x)\ dx \xrightarrow{\delta\to 0}$$

$$\sum_\beta \int_{\mathbb{R}^d} M_{\beta\alpha}1_{D^m_\alpha}(x)\psi(x,\theta_\beta,\nabla\tilde{v}_m(x)) =$$

$$= \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\ d(\boldsymbol{\nu}^\alpha_{t_m})(x,\theta,F).$$

Defined $\tilde{z}_\delta \colon D \to Z$ by $\tilde{z}_\delta(x) := \tilde{z}^\alpha_\delta(x)$ if $x \in D^m_\alpha$, we can conclude that $\boldsymbol{\delta}_{(\tilde{z}_\delta,\tilde{v}_m)} \rightharpoonup \boldsymbol{\nu}_{t_m}$ 2-weakly*, as $\delta \to 0$.

We observe that

$$\boldsymbol{\mu}_{t_1\ldots t_m} := \sum_{\alpha_1,\ldots,\alpha_{m-1},\alpha,\beta} M_{\beta\alpha}1_{D^1_{\alpha_1}}\cdot\dots\cdot 1_{D^{m-1}_{\alpha_{m-1}}}\cdot 1_{D^m_\alpha}\boldsymbol{\delta}_{(\theta_{\alpha_1},\ldots,\theta_{\alpha_{m-1}},\theta_\beta)} =$$

$$\sum_{\alpha\beta} M_{\beta\alpha}1_{D^m_\alpha}\boldsymbol{\delta}_{(z^1,\ldots,z^{m-1},\theta_\beta)},$$

where $z^j(x) := \theta_\gamma$ whenever $x \in D^j_\gamma$, for $j = 1,\ldots,m-1$. Hence we can apply the same argument used for $\boldsymbol{\nu}_{t_m}$ to $\boldsymbol{\mu}_{t_1\ldots t_m}$ and deduce that $\delta_{(z_1,\ldots,z_{m-1},\tilde{z}_\delta)} \rightharpoonup \boldsymbol{\mu}_{t_1\ldots t_m}$ weakly* as $\delta \to 0$. Hence it is enough to consider a sequence $\delta^k \to 0$, and, for every k, $z^k_j = z_j$ for $j < m$, $z^k_m = \tilde{z}_{\delta^k}$, $v^k_j = v_j$ for $j < m$, and $v^k_m = \tilde{v}_m$ to obtain the required approximation properties considered in Remark 5.3.2.

Consider now the case of $M_{\beta,\alpha}$ in $\mathcal{C}^1(D)$. Fixed a positive parameter ε, consider a finite family $(Q^i_\varepsilon)_{i=1}^{I(\varepsilon)}$ of disjoint cubes in $\mathbb{R}^d$, with diameter ε, covering D, and set

$$(M_{\beta\alpha})^i_\varepsilon := (M_{\beta\alpha})_{Q^i_\varepsilon\cap D} = \tfrac{1}{|Q^i_\varepsilon\cap D|}\int_{Q^i_\varepsilon\cap D} M_{\beta\alpha}(x)\ dx,$$

for every $i = 1,\ldots,I(\varepsilon)$, and every α, β. For a fixed α, we can define a measure $\boldsymbol{\nu}^\alpha_\varepsilon$ on $\mathbb{R}^d \times Z \times \mathbb{R}^{N\times d}$ by setting

$$\boldsymbol{\nu}^\alpha_\varepsilon := \sum_{i=1}^{I(\varepsilon)}\sum_{\beta=1}^{q}(M_{\beta\alpha})^i_\varepsilon 1_{Q^i_\varepsilon}1_{D^m_\alpha}\boldsymbol{\delta}_{(\theta_\beta,\nabla\tilde{v}_m)}.$$

Let us fix $i = 1,\ldots,I(\varepsilon)$ and reproduce the arguments used in the constant case: consider a measurable partition $((S^i_\varepsilon)^\alpha_\beta)_\beta$ of the unitary cube $[0,1]^d$, with $|(S^i_\varepsilon)^\alpha_\beta| = (M_{\beta\alpha})^i_\varepsilon$, for every β (it is possible to find such a partition since $\sum_\beta (M_{\beta\alpha})^i_\varepsilon = \frac{1}{|Q^i_\varepsilon\cap D|}\int_{Q^i_\varepsilon\cap D}(\sum_\beta M_{\beta\alpha})(x)\ dx = 1$, by the hypotheses on M), and define $(\tilde{z}^\alpha)^i_\varepsilon \colon \mathbb{R}^d \to Z$ as the 1-periodic measurable function satisfying

$$(\tilde{z}^\alpha)^i_\varepsilon(x) = \theta_\beta \quad \text{for a.e. } x \in (S^i_\varepsilon)^\alpha_\beta,$$

for every $\beta = 1, \dots, q$. For every $\delta > 0$, consider the function $(\tilde{z}^\alpha)^i_{\varepsilon,\delta}\colon \mathbb{R}^d \to Z$ defined by $(\tilde{z}^\alpha)^i_{\varepsilon,\delta}(x) := (\tilde{z}^\alpha)^i_\varepsilon(\frac{x}{\delta})$, for a.e. $x \in \mathbb{R}^d$. Fixed ε, we obtain as before that

$$1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde{z}^\alpha_{\varepsilon,\delta},\tilde{v}_m)} \rightharpoonup \boldsymbol{\nu}^\alpha_\varepsilon \quad \text{2-weakly*}, \tag{5.3.5}$$

as $\delta \to 0$, where $\tilde{z}^\alpha_{\varepsilon,\delta}\colon \mathbb{R}^d \to Z$ is the function defined by $\tilde{z}^\alpha_{\varepsilon,\delta} := \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(\tilde{z}^\alpha)^i_{\varepsilon,\delta}$.

Now we want to show that $\boldsymbol{\nu}^\alpha_\varepsilon \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ 2-weakly* as $\varepsilon \to 0$.

For every $\psi \in \mathcal{C}_0(D \times Z \times \mathbb{R}^{N\times d})$, we have

$$\begin{aligned}\Big|\int_{D\times Z\times\mathbb{R}^{N\times d}} &\psi(x,\theta,F)\; d\boldsymbol{\nu}^\alpha_{t_m}(x,\theta,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\; d\boldsymbol{\nu}^\alpha_\varepsilon(x,\theta,F)\Big| = \\ &= \Big|\sum_\beta \int_{D^m_\alpha} \Big[(M_{\beta\alpha}(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M_{\beta\alpha})^i_\varepsilon(x)\Big]\psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\; dx\Big|;\end{aligned}$$

Since for every $x \in D^m_\alpha$ there exists a unique $i_x = 1, \dots, I(\varepsilon)$ with $x \in Q^{i_x}_\varepsilon$, we have

$$|M_{\beta\alpha}(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M^n_{\beta\alpha})^i_\varepsilon| = |M^n_{\beta\alpha}(x) - (M_{\beta\alpha})^{i_x}_\varepsilon| \le \|\nabla M_{\beta\alpha}\|_\infty \varepsilon,$$

for every $x \in D^m_\alpha$ and every $\beta = 1, \dots, q$. Therefore we have

$$\begin{aligned}\Big|\int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\; d\boldsymbol{\nu}^\alpha_{t_m}(x,\theta,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\; d\boldsymbol{\nu}^\alpha_\varepsilon(x,\theta,F)\Big| \le \\ \le \sum_\beta \|\nabla M_{\beta\alpha}\|_\infty \|\psi\|_\infty |D|\varepsilon,\end{aligned} \tag{5.3.6}$$

which tends to 0 as $\varepsilon \to 0$.

Since $Y(D; Z \times \mathbb{R}^{N\times d})$ is contained in a bounded subset of the dual of a separable Banach space, it is metrizable with respect to the weak* topology. Let us denote by d a metric inducing on $Y(D; Z \times \mathbb{R}^{N\times d})$ the weak* topology, so that we have

- $d(\boldsymbol{\nu}^\alpha_\varepsilon, \boldsymbol{\nu}^\alpha_{t_m}) \to 0$ as $\varepsilon \to 0$;
- for every fixed ε, $d(1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde{z}^\alpha_{\varepsilon,\delta},\nabla\tilde{v})}, \boldsymbol{\nu}^\alpha_\varepsilon) \to 0$ as $\delta \to 0$.

Applying, as before, the same argument to $\boldsymbol{\mu}_{t_1\dots t_m}$, we deduce, using a diagonalization argument, that there exist sequences $\delta_k \to 0$ and $\varepsilon_k \to 0$ such that

(1) for every α, $1_{D^m_\alpha}\boldsymbol{\delta}_{(\tilde{z}^\alpha_{\varepsilon_k,\delta_k},\nabla v^m+\nabla\tilde{u})} \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ 2-weakly* as $k \to \infty$;

(2) for every α, we have

$$1_{D^m_\alpha}\boldsymbol{\delta}_{(z^1,\dots,z^m,\tilde{z}^\alpha_{\varepsilon_k,\delta_k})} \rightharpoonup \sum_{\alpha_1,\dots,\alpha_{m-1},\beta} M_{\beta\alpha} 1_{D^1_{\alpha_1}} \cdots 1_{D^{m-1}_{\alpha_{m-1}}} \cdot 1_{D^m_\alpha}\boldsymbol{\delta}_{(\theta_{\alpha_1},\dots,\theta_{\alpha_{m-1}},\theta_\beta)}$$

weakly*, as $k \to \infty$.

Now it is enough to define $\tilde{z}_{\varepsilon,\delta}\colon D \to Z$, by $\tilde{z}_{\varepsilon,\delta} := \sum_\alpha 1_{D^m_\alpha}\tilde{z}^\alpha_{\varepsilon,\delta}$, to prove the thesis.

It remains only to treat the general case of $M_{\beta\alpha} \in L^\infty(D)$. We can reproduce the same construction proposed in the $\mathcal{C}^1$-case; the only difference is that we have to use an approximation argument to show that $\boldsymbol{\nu}^\alpha_\varepsilon \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$. Indeed it is enough to consider, for every β, a sequence $(M^n_{\beta\alpha})_n$ in $\mathcal{C}^1(D)$, with $M^n_{\alpha\beta} \to M_{\beta\alpha}$ strongly in $L^1(D)$, as $n \to \infty$, and let $(M^n_{\beta\alpha})^i_\varepsilon := (M^n_{\beta\alpha})_{Q^i_\varepsilon \cap D}$. For every $\psi \in \mathcal{C}_0(D \times Z \times \mathbb{R}^{N\times d})$, we have

$$\Big| \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\ d\boldsymbol{\nu}^\alpha_{t_m}(x,\theta,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\ d\boldsymbol{\nu}^\alpha_\varepsilon(x,\theta,F)\Big| \le$$

$$\le \Big| \int_{D^m_\alpha} \sum_\beta M_{\beta\alpha}(x)\psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx - \int_{D^m_\alpha} \sum_\beta M^n_{\beta\alpha}(x)\psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx +$$

$$- \int_{D^m_\alpha} \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x) \sum_\beta (M_{\beta\alpha})^i_\varepsilon \psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx + \int_{D^m_\alpha} \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x) \sum_\beta (M^n_{\beta\alpha})^i_\varepsilon \psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx\Big| +$$

$$+\Big| \int_{D^m_\alpha} \sum_\beta M^n_{\beta\alpha}(x)\psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx - \int_{D^m_\alpha} \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x) \sum_\beta (M^n_{\beta\alpha})^i_\varepsilon \psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx\Big| \le$$

$$\le \Big| \sum_\beta \int_{D^m_\alpha} \Big[(M_{\beta\alpha} - M^n_{\beta\alpha})(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M_{\beta\alpha} - M^n_{\beta\alpha})^i_\varepsilon\Big] \psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx\Big| +$$

$$+\Big| \sum_\beta \int_{D^m_\alpha} \Big[(M^n_{\beta\alpha}(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M^n_{\beta\alpha})^i_\varepsilon(x)\Big] \psi(x,\theta_\beta,\nabla\tilde{v}_m(x))\ dx\Big|.$$

We know that

$$|M^n_{\beta\alpha}(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M^n_{\beta\alpha})^i_\varepsilon| = |M^n_{\beta\alpha}(x) - (M^n_{\beta\alpha})^{i_x}_\varepsilon| \le \|\nabla M^n_{\beta\alpha}\|_\infty \varepsilon,$$

for every $x \in D^m_\alpha$ and every $\beta = 1, \dots, q$. On the other hand, using Lemma 1.1.1, we can deduce that

$$\int_{D^m_\alpha} |(M_{\beta\alpha} - M^n_{\beta\alpha})(x) - \sum_{i=1}^{I(\varepsilon)} 1_{Q^i_\varepsilon}(x)(M_{\beta\alpha} - M^n_{\beta\alpha})^i_\varepsilon)|\ dx \le \int_{D^m_\alpha} |M_{\beta\alpha} - M^n_{\beta\alpha}(x)|\ dx.$$

Let us fix $\eta > 0$; choosing $\bar{n}$ such that $\sum_\beta \|M_{\beta\alpha} - M^{\bar{n}}_{\beta\alpha}\|_1 \|\psi\|_\infty \le \eta/2$, we have

$$\Big| \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\ d\boldsymbol{\nu}^\alpha_{t_m}(x,\theta,F) - \int_{D\times Z\times\mathbb{R}^{N\times d}} \psi(x,\theta,F)\ d\boldsymbol{\nu}^\alpha_\varepsilon(x,\theta,F)\Big| \le \eta,$$

for every $\varepsilon \le \varepsilon_\eta := \eta(2\sum_\beta \|\nabla M^{\bar{n}}_{\beta\alpha}\|_\infty \|\psi\|_\infty |D|)^{-1}$; therefore we obtain that $\boldsymbol{\nu}^\alpha_\varepsilon \rightharpoonup \boldsymbol{\nu}^\alpha_{t_m}$ as $\varepsilon \to 0$ and we can prove the thesis as in the previous case.

□

REMARK 5.3.5. If $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(A, q, \boldsymbol{w})$, for every $t \in A$ there exists a unique function $\boldsymbol{v}(t) \in \boldsymbol{w}(t) + H^1_0(D;\mathbb{R}^N)$ such that $\nabla \boldsymbol{v}(t) = \sum_\alpha \boldsymbol{b}^t_\alpha \mathrm{bar}(\boldsymbol{\lambda}^t_\alpha)$; moreover, for every $t \in A$, the function $\boldsymbol{\sigma}(t)$ representing the *stress* and defined by

$$\boldsymbol{\sigma}(t,x) := \sum_{\alpha=1}^q \boldsymbol{b}^t_\alpha(x) \int_{\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F) \quad \text{for a.e. } x \in D$$

belongs to $L^2(D;\mathbb{R}^{N\times d})$.

5.4. Definition of quasistatic evolution and main result

First of all we give the definition of quasistatic evolution in the discrete setting.

DEFINITION 5.4.1. Given $\boldsymbol{\varphi} \in AC([0,T]; W^{1,p}(D;\mathbb{R}^N))$, for $2 < p < +\infty$, $T > 0$, $z_0 \in L^\infty(D;Z)$, and $v_0 \in \mathcal{A}(0)$, a *quasistatic evolution of Young measures* with boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0, v_0), in the time interval $[0,T]$, is a pair $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad([0,T], q, \boldsymbol{\varphi})$, with $\boldsymbol{b} \in S_-([0,T], D, q)$, satisfying the following conditions:

(ev0) *initial condition*: defined $D^0_\alpha := \{x \in D \ : \ z_0(x) = \theta_\alpha\}$, we have $\boldsymbol{b}^0_\alpha = 1_{D^0_\alpha}$ and $(\boldsymbol{\lambda}^0_\alpha)^x = \delta_{\nabla v_0(x)}$ if $x \in D^0_\alpha$, for every α;

(ev1) *partial-global stability*: for every $t \in [0,T]$, we have

$$\sum_{\alpha=1}^q \int_D \boldsymbol{b}^t_\alpha(x) \big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F) \big)\ dx \leq$$

$$\leq \sum_{\alpha,\beta=1}^q \int_D M_{\beta\alpha}(x) \boldsymbol{b}^t_\alpha(x) \big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta, F + \nabla \tilde{u}(x))\ d(\boldsymbol{\lambda}^t_\alpha)^x(F) \big)\ dx +$$

$$+ \sum_{\alpha,\beta=1}^q H(\theta_\beta, \theta_\alpha) \int_D M_{\beta\alpha}(x) \boldsymbol{b}^t_\alpha(x)\ dx,$$

for every $\tilde{u} \in H^1_0(D;\mathbb{R}^N)$, and every measurable map

$$M\colon D \to \mathbb{M}^{q\times q}_{St} \qquad x \mapsto (M_{\beta\alpha}(x))_{\beta\alpha};$$

(ev2) *energy inequality*: if $\boldsymbol{\sigma}$ is the function defined in Remark (5.3.5), then the map

$$t \mapsto \langle \boldsymbol{\sigma}(t), \nabla \dot{\boldsymbol{\varphi}}(t) \rangle \tag{5.4.1}$$

is integrable on $[0,T]$, and for every $t_1 < t_2 \in [0,T]$ we have

$$\sum_{\alpha=1}^q \int_D \boldsymbol{b}^{t_2}_\alpha(x) \big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^{t_2}_\alpha)^x(F) \big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) \leq$$

$$\leq \sum_{\alpha=1}^q \int_D \boldsymbol{b}^{t_1}_\alpha(x) \big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^{t_1}_\alpha)^x(F) \big)\ dx + \int_{t_1}^{t_2} \langle \boldsymbol{\sigma}(s), \nabla \dot{\boldsymbol{\varphi}}(s) \rangle\ ds, \tag{5.4.2}$$

where $\mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2)$ is defined by (2.3.6).

THEOREM 5.4.2. *Let $\boldsymbol{\varphi} \in AC([0,T]; H^1(D;\mathbb{R}^N))$ and $T > 0$. Assume that the partial-global stability condition is satisfied by $(z_0, v_0) \in L^\infty(D;Z) \times \mathcal{A}(0)$. Then there exists a quasistatic evolution of Young measures with boundary datum $\boldsymbol{\varphi}$ and initial condition (z_0, v_0) in the time interval $[0,T]$.*

5.5. Proof of the main theorem

The proof is obtained via time-discretization, resolution of incremental minimum problems, and passing to the limit.

5.5.1. The incremental minimum problem. The first step of the proof consists in the definition of an approximate solution via an inductive minimization process.

Let us fix a sequence of subdivisions of $[0,T]$, $0 = t_n^0 < t_n^1 < \cdots < t_n^{k(n)} = T$, such that $\sup_{i=1,\ldots,k(n)} \tau_n^i \to 0$, as $n \to \infty$, where $\tau_n^i := t_n^i - t_n^{i-1}$, for every $i = 1, \ldots, k(n)$.

For every $i = 0, 1, \ldots, k(n)$ we set $\varphi_n^i := \boldsymbol{\varphi}(t_n^i)$.

We define $(\boldsymbol{b}_n^i, \boldsymbol{\lambda}_n^i) \in Ad(\{t_n^0, \ldots, t_n^i\}, q, \boldsymbol{\varphi})$ by induction on i: set

$$(\boldsymbol{b}_n^0)_\alpha (\bar{\boldsymbol{\lambda}}_n^0)_\alpha := 1_{D_\alpha^0} \boldsymbol{\delta}_{\nabla v_0},$$

where $D_\alpha^0 := \{x \in D \ : \ z_0(x) = \theta_\alpha\}$; for $i > 0$ we define $(\boldsymbol{b}_n^i, \boldsymbol{\lambda}_n^i)$ as a pair satisfying the following properties:

(min) $(\boldsymbol{b}_n^i, \boldsymbol{\lambda}_n^i)$ is a minimizer of the functional

$$\sum_\alpha \int_D \boldsymbol{b}_\alpha^{t_n^i}(x) \Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F) \ d(\boldsymbol{\lambda}_\alpha^{t_n^i})^x(F) \Big) \ dx + \\ + \sum_{\alpha,\beta} H(\theta_\beta, \theta_\alpha) \int_D \boldsymbol{b}_{\alpha\beta}^{t_n^{i-1} t_n^i}(x) \ dx, \tag{5.5.1}$$

over the set $A_n^i(\boldsymbol{b}_n^{i-1}, \boldsymbol{\lambda}_n^{i-1})$ of all $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(\{t_n^0, \ldots, t_n^i\}, q, \boldsymbol{\varphi})$ satisfying

$$\sum_\beta \boldsymbol{b}_{\alpha_0 \ldots \alpha_{i-1}\beta}^{t_n^0 \ldots t_n^i} = (\boldsymbol{b}_n^{i-1})_{\alpha_0 \ldots \alpha_{i-1}}^{t_n^0 \ldots t_n^{i-1}} \quad \text{a.e. in } D, \text{ for every } (\alpha_0, \ldots, \alpha_{i-1}) \in \mathscr{A}_i^q \tag{5.5.2}$$

$$\boldsymbol{\lambda}_\alpha^{t_n^j} = (\boldsymbol{\lambda}_n^{i-1})_\alpha^{t_n^j}, \quad \text{for every } j < i \text{ and every } \alpha; \tag{5.5.3}$$

(reg) there exist constants $r > 1$ and $\gamma > 0$, both independent of i and n, such that

$$\sum_{\alpha=1}^q \int_{D \times \mathbb{R}^{N\times d}} (\boldsymbol{b}_n^i)_\alpha^{t_n^i}(x) |F|^{2r} \ d(\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i}(x, F) \le \\ \le \gamma \Big[1 + \Big(\sum_{\alpha=1}^q \int_{D \times \mathbb{R}^{N\times d}} (\boldsymbol{b}_n^i)_\alpha^{t_n^i}(x) |F|^2 \ d(\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i}(x, F) \Big)^r \Big]. \tag{5.5.4}$$

The existence of such a pair $(\boldsymbol{b}_n^i, \boldsymbol{\lambda}_n^i)$ is proved in Lemma 5.5.2 below.

LEMMA 5.5.1. *For every $i > 1$ and every $(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1}) \in Ad(\{t_n^0, \ldots, t_n^{i-1}\}, q, \boldsymbol{\varphi})$, the set $A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$ is nonempty.*

PROOF. Fixed $(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1}) \in Ad(\{t_n^0, \ldots, t_n^{i-1}\}, q, \boldsymbol{\varphi})$, we consider the map

$$\tilde{\mathcal{T}}^2_{\nabla\varphi_n^i - \nabla\varphi_n^{i-1}} \colon (x, F) \mapsto (x, F + \nabla\varphi_n^i(x) - \nabla\varphi_n^{i-1}(x));$$

let $\boldsymbol{b}$ be the unique element of $S(\{t_n^0, \ldots, t_n^i\}, D, q)$ satisfying

$$\boldsymbol{b}_{\alpha_0 \ldots \alpha_{i-1}\alpha_i}^{t_n^0 \ldots t_n^{i-1} t_n^i} := \begin{cases} (\boldsymbol{b}^{i-1})_{\alpha_0 \ldots \alpha_{i-1}}^{t_n^0 \ldots t_n^{i-1}} & \text{if } \alpha_i = \alpha_{i-1} \\ 0 & \text{otherwise;} \end{cases} \tag{5.5.5}$$

for every $(\alpha_0, \ldots, \alpha_i) \in \mathscr{A}_{i+1}^q$; define $\boldsymbol{\lambda} \in ((Y(D; \mathbb{R}^{N\times d})^q)^{\{t_n^0, \ldots, t_n^i\}}$ by

$$\boldsymbol{\lambda}_\alpha^{t_n^j} := (\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^j} \quad \text{if } j < i, \text{ for every } \alpha, \\ \boldsymbol{\lambda}_\alpha^{t_n^i} := \tilde{\mathcal{T}}^2_{\nabla\varphi_n^i - \nabla\varphi_n^{i-1}}((\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^{i-1}}). \tag{5.5.6}$$

It is immediate to see that $(\boldsymbol{b},\boldsymbol{\lambda})$ satisfy the properties (5.5.2) and (5.5.3). By construction $\boldsymbol{b}_\alpha^{t_n^j}\boldsymbol{\lambda}_\alpha^{t_n^j} \in Y^2(D;\mathbb{R}^{N\times d})$, for every α and every $j=0,\dots,i$: indeed for $j<i$ it is obvious, while for i we have

$$\boldsymbol{b}_\alpha^{t_n^i} = \sum_{(\alpha_0\dots\alpha_{i-1})} \boldsymbol{b}_{\alpha_0\dots\alpha_{i-1}\alpha}^{t_n^0\dots t_n^{i-1}t_n^i} =$$
$$\sum_{(\alpha_0\dots\alpha_{i-2})} (\boldsymbol{b}^{i-1})_{\alpha_0\dots\alpha_{i-2}\alpha}^{t_n^0\dots t_n^{i-1}} = (\boldsymbol{b}^{i-1})_\alpha^{t_n^{i-1}},$$

for every α, therefore

$$\int_D \boldsymbol{b}_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}} |F|^2\, d(\boldsymbol{\lambda}_\alpha^{t_n^i})^x(F)\Big)\, dx =$$
$$= \int_D (\boldsymbol{b}^{i-1})_\alpha^{t_n^{i-1}}(x)\Big(\int_{\mathbb{R}^{N\times d}} |F+\nabla\varphi_n^i(x)-\nabla\varphi_n^{i-1}(x)|^2\, d((\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^{i-1}})^x(F)\Big)\, dx \leq$$
$$\leq \int_D (\boldsymbol{b}^{i-1})_\alpha^{t_n^{i-1}}(x)\Big(\int_{\mathbb{R}^{N\times d}} |F|^2\, d((\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^{i-1}})^x(F)\Big)\, dx + \|\nabla\varphi_n^i\|_2^2 + \|\nabla\varphi_n^{i-1}\|_2^2 < +\infty,$$

for every α. It is now easy to prove the approximations properties (1) and (2) of Definition 5.3.1 for $(\boldsymbol{b},\boldsymbol{\lambda})$ defined by (5.5.5) and (5.5.6). Suppose that for every k and every $j = 0,\dots,i-1$, $((D^{i-1})_\alpha^{j,k})_\alpha$ is a measurable partition of D and $(v^{i-1})^{j,k}$ is a function in $\varphi_n^j + H_0^1(D;\mathbb{R}^N)$, which satisfy conditions (1) and (2) for $(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$. Then $(D_\alpha^{j,k})_\alpha$ and $v^{j,k}$, defined by

$$D_\alpha^{j,k} := (D^{i-1})_\alpha^{j,k} \quad \text{for } j<i$$
$$D_\alpha^{i,k} := (D^{i-1})_\alpha^{i-1,k}$$
$$v^{j,k} := (v^{i-1})^{j,k} \quad \text{for } j<i$$
$$v^{i,k} := (v^{i-1})^{i-1,k} + \varphi_n^i - \varphi_n^{i-1},$$

for every α and every k, satisfy (1) and (2) for $(\boldsymbol{b},\boldsymbol{\lambda})$. □

LEMMA 5.5.2. *There exist constants $\gamma>0$ and $r>1$, such that for every n, every $i=1,\dots,k(n)$, and every $(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1}) \in Ad(\{t_n^0,\dots,t_n^{i-1},q,\boldsymbol{\varphi})$, the functional (5.5.1) has a minimizer over $A_n^i(\boldsymbol{b}^{i-1},\boldsymbol{\lambda}^{i-1})$, which satisfies (5.5.4).*

PROOF. Let $(\boldsymbol{b}^h,\boldsymbol{\lambda}^h)_h$ be a minimizing sequence. By the bounds on W we have

$$c\sum_\alpha \int_D (\boldsymbol{b}^h)_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}} |F|^2\, d((\boldsymbol{\lambda}^h)_\alpha^{t_n^i})^x(F)\Big)\, dx - C \leq \tag{5.5.7}$$
$$\leq \sum_\alpha \int_D (\boldsymbol{b}^h)_\alpha^{t_n^i}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha,F)\, d((\boldsymbol{\lambda}^h)_\alpha^{t_n^i})^x(F)\Big) \leq C',$$

for every h, for a positive constant C' independent of h. Moreover, we have

$$\sup_h \|(\boldsymbol{b}^h)_{\alpha_0\dots\alpha_i}^{t_n^0\dots t_n^i}\|_\infty \leq 1,$$

for every $(\alpha_0,\dots,\alpha_i) \in \mathscr{A}_{i+1}^q$. Therefore, we can deduce that there exist $(b_{(\alpha_0\dots\alpha_i)})_{(\alpha_0\dots\alpha_i)} \in (L^\infty(D;[0,1]))^{q^{i+1}}$ satisfying (2.3.3) and

$$(\boldsymbol{b}^h)_{\alpha_0\dots\alpha_i}^{t_n^0\dots t_n^i} \rightharpoonup b_{\alpha_0\dots\alpha_i} \quad L^\infty\text{-weakly*},$$

as $h \to \infty$, up to a subsequence. In particular,

$$(\boldsymbol{b}^h)_\alpha^{t_n^i} \rightharpoonup \sum_{(\alpha_0 \dots \alpha_{i-1})} \quad L^\infty\text{-weakly*}. \tag{5.5.8}$$

From (5.5.7), we can deduce using Remark 2.3.6 and (5.5.8) that there exists $\lambda \in Y(D; \mathbb{R}^{N\times d})^q$ satisfying with $\sum_{(\alpha_0,\dots,\alpha_{i-1})} b_{\alpha_0\dots\alpha_{i-1}\alpha}$ condition (2.3.10)(with $p = 2$), such that

$$(\boldsymbol{b}^h)_\alpha^{t_n^i}(\boldsymbol{\lambda}^h)_\alpha^{t_n^i} \rightharpoonup \sum_{(\alpha_0 \dots \alpha_{i-1})} b_{\alpha_0\dots\alpha_{i-1}\alpha}\lambda_\alpha \quad 2\text{-weakly*},$$

as $h \to \infty$, up to a subsequence.

We now define

$$\boldsymbol{\lambda}_\alpha^{t_n^j} := (\boldsymbol{\lambda}^{i-1})_\alpha^{t_n^j}, \quad \text{for every } j < i \text{ and every } \alpha,$$
$$\boldsymbol{\lambda}_\alpha^{t_n^i} := \lambda_\alpha \quad \text{for every } \alpha,$$

and $\boldsymbol{b}$ as the unique element in $S(\{t_n^0, \dots, t_n^i\}, D, q)$ satisfying

$$\boldsymbol{b}_{\alpha_0\dots\alpha_i}^{t_n^0\dots t_n^i} := b_{\alpha_0\dots\alpha_i} \quad \text{for every } (\alpha_0 \dots \alpha_i) \in \mathscr{A}_{i+1}^q.$$

It is immediate to see that the hypotheses of Lemma 5.3.3 are satisfied by $(\boldsymbol{b}, \boldsymbol{\lambda})$, hence $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad(\{t_n^0, \dots, t_n^i\}, q, \boldsymbol{\varphi})$. Moreover $(\boldsymbol{b}, \boldsymbol{\lambda})$ satisfies (5.5.2) and (5.5.3), by construction; hence $(\boldsymbol{b}, \boldsymbol{\lambda}) \in A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$.

The term of (5.5.1) containing W is 2-weakly* lower semicontinuous, while the one containing H is L^∞-weakly* continuous; therefore, the functional (5.5.1) is lower semicontinuous with respect to the convergence we are considering, this implies that $(\boldsymbol{b}, \boldsymbol{\lambda})$ is a solution of our minimum problem.

Now we want to construct from $(\boldsymbol{b}, \boldsymbol{\lambda})$ a new minimizer $(\boldsymbol{b}, \bar{\boldsymbol{\lambda}})$ satisfying property (5.5.4). Let us set

$$(\boldsymbol{\nu}_n^i)_{t_n^i} := \sum_{\alpha=1}^q (\boldsymbol{b}_n^i)_\alpha^{t_n^i}(\delta_{\theta_\alpha} \otimes (\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i}),$$
$$(\boldsymbol{\mu}_n^i)_{t_n^0\dots t_n^i} := \sum_{(\alpha_0,\dots,\alpha_i)} (\boldsymbol{b}_n^i)_{\alpha_0\dots\alpha_i}^{t_n^0\dots t_n^i}\boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_i})}.$$

From the definition of $A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$ it follows that $(\boldsymbol{\nu}_n^i, \boldsymbol{\mu}_n^i) \in AY(\{t_n^0, \dots, t_n^i\}, Z, \boldsymbol{\varphi})$; in particular there exist sequences $(z_{n,k}^{i-1})_k$, $((z_{n,k}^i)_k$ in $L^\infty(D; Z)$, and $(v_{n,k}^i)_k$ in $\mathcal{A}(t_n^i)$ satisfying

$$\boldsymbol{\delta}_{(z_{n,k}^{i-1}, z_{n,k}^i)} \rightharpoonup (\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i} \quad \text{weakly*},$$
$$\boldsymbol{\delta}_{(z_{n,k}^i, \nabla v_{n,k}^i)} \rightharpoonup (\boldsymbol{\nu}_n^i)_{t_n^i} \quad 2\text{-weakly*},$$

as $k \to \infty$. Thanks to Lemma 1.3.8, we can assume, without loss of generality, that $(|\nabla v_{n,k}^i|^2)_k$ are equiintegrable; hence by Theorem 1.3.10 we may assume that

$$\sup_k \|\nabla v_{n,k}^i\|_2^2 \le \int_{D\times Z\times\mathbb{R}^{N\times d}} |F|^2 \; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F) + 1, \tag{5.5.9}$$

$$\int_D W(z_{n,k}^i(x), \nabla v_{n,k}^i(x)) \; dx \to \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta, F) \; d(\boldsymbol{\nu}_n^i)_{t_n^i}(x, \theta, F). \tag{5.5.10}$$

Denote by I_n^i the minimum value of (5.5.1) over $A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$. Thanks to (5.5.10), we can deduce that

$$\lim_k \Big[\int_D W(z_{n,k}^i(x), \nabla v_{n,k}^i(x))\ dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\ dx \Big] =$$
$$= \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta, F)\ d(\boldsymbol{\nu}_n^i)_{t_n^i}(x,\theta,F) +$$
$$+ \int_{D\times Z^2} H(\theta_i, \theta_{i-1})\ d(\boldsymbol{\mu}_n^i)_{t_n^{i-1} t_n^i}(x, \theta_{i-1}, \theta_i) = I_n^i. \tag{5.5.11}$$

Now we want to consider the following auxiliary minimum problem, for every k:

$$I_{n,k}^i := \inf_{v\in\varphi_n^i + H_0^1} \int_D W(z_{n,k}^i(x), \nabla v(x))\, dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\, dx. \tag{5.5.12}$$

For every k, we choose $\hat{v}_{n,k}^i \in \varphi_n^i + H_0^1(D;\mathbb{R}^{N\times d})$ such that

$$\int_D W(z_{n,k}^i(x), \nabla \hat{v}_{n,k}^i(x))\, dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\, dx \le I_{n,k}^i + \tfrac{1}{k}. \tag{5.5.13}$$

Using $v_{n,k}^i$ as competitor in (5.5.12), we can easily deduce, from (5.5.13) and the growth hypothesis on W, that

$$\|\nabla \hat{v}_{n,k}^i\|_2^2 \le \hat{C}(1 + \|\nabla v_{n,k}^i\|_2^2),$$

for a suitable positive constant $\hat{C}$, independent of n. Hence, thanks to (5.5.9), $\sup_k \|\nabla \hat{v}_{n,k}^i\|_2^2$ is bounded; in particular there exists $\bar{\nu}_n^i \in Y^2(D; Z\times\mathbb{R}^{N\times d})$ such that, up to a subsequence, $\boldsymbol{\delta}_{(z_{n,k}^i, \nabla\hat{v}_{n,k}^i)} \rightharpoonup \bar{\nu}_n^i$ 2-weakly* as $k\to\infty$. Thanks to Lemma 1.3.8 we can assume, up to a subsequence, that $|\nabla \hat{v}_{n,k}^i|^2$ is equiintegrable in k.

Since $\pi_{D\times Z}(\bar{\nu}_n^i) = \sum_\alpha (\boldsymbol{b}_n^i)_\alpha^{t_n^i} \boldsymbol{\delta}_{\theta_\alpha}$, by Remark 2.3.4 there exists a family of Young measures $\bar{\lambda}_n^i = ((\bar{\lambda}_n^i)_\alpha)_\alpha$ such that it holds

$$\bar{\nu}_n^i = \sum_{\alpha=1}^q (\boldsymbol{b}_n^i)_\alpha^{t_n^i} (\delta_{\theta_\alpha} \otimes (\bar{\lambda}_n^i)_\alpha); \tag{5.5.14}$$

since $\bar{\nu}_n^i \in Y^2(D; Z\times\mathbb{R}^{N\times d})$, $(\boldsymbol{b}_n^i)_\alpha^{t_n^i} (\bar{\lambda}_n^i)_\alpha$ satisfies (2.3.10) for $p=2$. We have

$$\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta, F)\ d\bar{\nu}_n^i(x,\theta,F) + \int_{D\times Z^2} H(\theta_i, \theta_{i-1})\ d(\boldsymbol{\mu}_n^i)_{t_n^{i-1} t_n^i}(x, \theta_{i-1}, \theta_i) \le$$
$$\le \liminf_k \Big[\int_D W(z_{n,k}^i(x), \nabla \hat{v}_{n,k}^i(x))\ dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\ dx \Big] \le$$
$$\le \liminf_k [I_{n,k}^i + 1/k] \le \tag{5.5.15}$$
$$\le \liminf_k \Big[\int_D W(z_{n,k}^i(x), \nabla v_{n,k}^i(x))\ dx + \int_D H(z_{n,k}^i(x), z_{n,k}^{i-1}(x))\ dx \Big] = I_n^i.$$

The construction of $\bar{\nu}_n^i$ implies that the pair $(\boldsymbol{b}, \bar{\boldsymbol{\lambda}})$, with

$$\bar{\boldsymbol{\lambda}} := (\boldsymbol{\lambda}^{t_n^0}, \dots, \boldsymbol{\lambda}^{t_n^{i-1}}, \bar{\lambda}_n^i) =$$
$$= ((\boldsymbol{\lambda}^{i-1})^{t_n^0}, \dots, (\boldsymbol{\lambda}^{i-1})^{t_n^{i-1}}, \bar{\lambda}_n^i),$$

is an element of $Ad(\{t_n^0, \dots, t_n^i\}, q, \boldsymbol{\varphi})$; moreover it satisfies the "memory properties" (5.5.2) and (5.5.3), required to be in $A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$. Hence

$$I_n^i \leq \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta, F)\; d\bar{\nu}_n^i(x,\theta,F) + \int_{D\times Z^2} H(\theta_i, \theta_{i-1})\; d(\boldsymbol{\mu}_n^i)_{t_n^{i-1}t_n^i}(x, \theta_{i-1}, \theta_i); \tag{5.5.16}$$

we can deduce from (5.5.15) and (5.5.16) that $(\boldsymbol{b}, \bar{\boldsymbol{\lambda}})$ minimizes (5.5.1) over $A_n^i(\boldsymbol{b}^{i-1}, \boldsymbol{\lambda}^{i-1})$.

Now we want to apply Ekeland Principle in order to construct a more regular sequence $(\bar{v}_{n,k}^i)_k$ which, together with $z_{n,k}^i$, generates $\bar{\nu}_n^i$.

We define $\hat{u}_{n,k}^i$ as the function $\hat{v}_{n,k}^i - \varphi_n^i \in H_0^1(D;\mathbb{R}^N)$. Consider the functional $\mathcal{E}$ defined on the Banach space $W_0^{1,1}(D;\mathbb{R}^N)$ by

$$\mathcal{E}(u) := \begin{cases} \int_D W(z_{n,k}^i(x), \nabla\varphi_n^i(x) + \nabla u(x))\; dx & \text{if } u \in H_0^1(D;\mathbb{R}^N); \\ +\infty & \text{otherwise;} \end{cases}$$

This functional is strongly lower semicontinuous with respect to the $W_0^{1,1}$ topology, it is positive and not infinite everywhere: hence we apply Ekeland's Principle (see [**16**, Corollary 6.1, p.30]) to $W_0^{1,1}(D;\mathbb{R}^N)$ endowed with the norm $\|u\|_{W_0^{1,1}} := \|\nabla u\|_1$; we deduce that there exists $\bar{u}_{n,k}^i \in H_0^1(D;\mathbb{R}^N)$ satisfying the following properties:

$$\int_D W(z_{n,k}^i(x), \nabla\varphi_n^i(x) + \nabla\bar{u}_{n,k}^i(x))\; dx \leq I_{n,k}^i + \frac{1}{k}; \tag{5.5.17}$$

$$\|\nabla\bar{u}_{n,k}^i - \nabla\hat{u}_{n,k}^i\|_1 \leq \frac{1}{\sqrt{k}}; \tag{5.5.18}$$

$$\int_D W(z_{n,k}^i(x), \nabla\varphi_n^i(x) + \nabla\bar{u}_{n,k}^i(x))\; dx \leq \tag{5.5.19}$$

$$\int_D \Big[W(z_{n,k}^i(x), \nabla\varphi_n^i(x) + \nabla u(x)) + \frac{1}{\sqrt{k}}|\nabla u - \nabla\bar{u}_{n,k}^i|\Big]\; dx,$$

for every $u \in H_0^1(D;\mathbb{R}^N)$.

In particular these properties imply that

$$\sup_k \|\nabla\bar{u}_{n,k}^i\|_2^2 \leq \bar{C}(1 + \sup_k \|\nabla\hat{u}_{n,k}^i\|_2^2), \tag{5.5.20}$$

for a suitable positive constant $\bar{C}$ independent of k, n, and i, and

$$\boldsymbol{\delta}_{(z_{n,k}^i, \nabla\varphi_n^i + \nabla\bar{u}_{n,k}^i)} \rightharpoonup \bar{\nu}_n^i \quad \text{2-weakly*},$$

as $k \to \infty$.

Using the growth hypotheses on W, it is easy to deduce from (5.5.19) that, for k sufficiently large, $\bar{v}_{n,k}^i$ is a Q-quasi-minimum of the functional $\mathcal{F}\colon H^1(D;\mathbb{R}^N) \to \mathbb{R}$ defined by $\mathcal{F}(v) = \int_D (1+|\nabla v(x)|^2)\; dx$, for a suitable positive constant Q independent of k, n, and i. We can now apply Theorem 1.2.4, and conclude that there exist two constants $\gamma > 0$ and $r > 1$, both independent of k, n, and i, such that

$$\int_D |\nabla\bar{v}_{n,k}^i(x)|^{2r}\; dx \leq \gamma\Big[1 + \big(\int_D |\nabla\bar{v}_{n,k}^i(x)|^2\; dx\big)^r\Big],$$

for every k. In particular, thanks to (5.5.20), we have

$$\int_D |\nabla \bar{v}^i_{n,k}(x)|^{2r}\ dx \le \gamma\Big[1 + \|\nabla \bar{v}^i_{n,k}\|_2^{2r}\Big] \le$$
$$\le \tilde{\gamma}\Big[(1 + \|\nabla \hat{v}^i_{n,k}\|_2^{2r}\Big], \tag{5.5.21}$$

for a suitable constant $\tilde{\gamma} > 0$ independent of k, n, and i. Thanks to the equiintegrability of $|\nabla \hat{v}^i_{n,k}|^2$, using Theorem 1.3.10 we can deduce that

$$\sum_{\alpha=1}^{q} \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^{t^i_n}_\alpha(x)|F|^{2r}\ d\bar{\lambda}^i_n(x,F) \le$$
$$\le \liminf_k \int_D |\nabla \bar{v}^i_{n,k}(x)|^{2r}\ dx \le$$
$$\le \tilde{\gamma}[1 + (\lim_k \int_D |\nabla \hat{v}^i_{n,k}|^2\ dx)^r] =$$
$$= \tilde{\gamma}[1 + (\sum_\alpha \int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^{t^i_n}_\alpha(x)|F|^2\ d\bar{\lambda}^i_n(x,F))^r].$$

This concludes the proof. □

Using the minimization process described so far, it is possible to construct inductively $(\boldsymbol{b}^i_n, \boldsymbol{\lambda}^i_n)$, for every $i = 1, \dots, k(n)$ and every n.

Set $\tau^n(s) := t^i_n$, whenever $t^i_n \le s < t^{i+1}_n$, where we set $t^{k(n)+1}_n := T + \frac{1}{n}$.

For every i and n we set

$$\sigma^i_n(x) := \sum_{\alpha=1}^{q} \int_{\mathbb{R}^{N\times d}} (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x) \frac{\partial W}{\partial F}(\theta_\alpha, F)\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F),$$

and define

$$\boldsymbol{\sigma}_n(t,x) := \sigma^i_n(x), \tag{5.5.22}$$

for a.e. $x \in D$, whenever $t^i_n \le t < t^{i+1}_n$.

For every $\alpha = 1, \dots, q$, we define $(\boldsymbol{\lambda}_n)_\alpha \in Y(D;\mathbb{R}^{N\times d})^{[0,T]}$ by

$$(\boldsymbol{\lambda}_n)^s_\alpha := (\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha, \tag{5.5.23}$$

whenever $t^i_n = \tau^n(s)$, for every $s \in [0,T]$; we define also $\boldsymbol{b}_n \in S([0,T], D; \mathbb{R}^m)$ as the piecewise constant interpolation of $\boldsymbol{b}^{k(n)}_n$ (see Definition 2.3.1).

Note that $(\boldsymbol{b}_n, \boldsymbol{\lambda}_n) \in Ad([0,T], q, \boldsymbol{\varphi}(\tau^n(\cdot)))$ by construction.

5.5.2. A priori estimates.

Set

$$(\boldsymbol{\nu}^i_n)_{t^i_n} := \sum_{\alpha=1}^{q} (\boldsymbol{b}^i_n)^{t^i_n}_\alpha (\delta_{\theta_\alpha} \otimes (\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha),$$
$$(\boldsymbol{\mu}^i_n)_{t^0_n\dots t^i_n} := \sum_{(\alpha_0,\dots,\alpha_i)} (\boldsymbol{b}^i_n)^{t^0_n\dots t^i_n}_{\alpha_0\dots\alpha_i} \boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_i})},$$

for every $i = 1, \dots, k(n)$, and

$$(\boldsymbol{\nu}_n)_t = \sum_{\alpha=1}^{q} (\boldsymbol{b}_n)^t_\alpha (\delta_{\theta_\alpha} \otimes (\boldsymbol{\lambda}_n)^t_\alpha),$$
$$(\boldsymbol{\mu}_n)_{t_0 \dots t_m} := \sum_{(\alpha_0,\dots,\alpha_m)} (\boldsymbol{b}_n)^{t_0\dots t_m}_{\alpha_0\dots\alpha_n} \boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_m})},$$

for every $t \in [0,T]$ and every $t_0 < \cdots < t_m$ in $[0,T]$.

As in Section 3.4.2, we want to deduce a discrete version of the energy inequality for $(\boldsymbol{b}_n, \boldsymbol{\lambda}_n)$. We briefly recall the arguments for the reader's convenience.

Using the competitor defined in the proof of Lemma 5.5.1, we have

$$\begin{aligned} &\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}^i_n)_{t^i_n}(x,\theta,F) + \\ &+ \int_{D\times(Z)^2} H(\theta_i - \theta_{i-1})\; d(\boldsymbol{\mu}^i_n)_{t^{i-1}_n t^i_n}(x,\theta_{i-1},\theta_i) \le \\ &\le \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}^{i-1}_n)_{t^{i-1}_n}(x,\theta,F) + \\ &+ \int_{D\times Z\times\mathbb{R}^{N\times d}} [W(\theta, F + \nabla\varphi^i_n(x) - \nabla\varphi^{i-1}_n(x)) - W(\theta,F)]\; d(\boldsymbol{\nu}^{i-1}_n)_{t^{i-1}_n}(x,\theta,F). \end{aligned}$$

Let us fix $t_1 < t_2$ in $[0,T]$ such that $t^l_n \le t_1 < t^{l+1}_n \le t^j_n \le t_2 < t^{j+1}_n$, for suitable $l, j = 0, \dots, k(n)$; using

$$\begin{aligned} &\int_{D\times Z\times\mathbb{R}^{N\times d}} [W(\theta, F + \nabla\varphi^i_n(x) - \nabla\varphi^{i-1}_n(x)) - W(\theta,F)]\; d(\boldsymbol{\nu}^{i-1}_n)_{t^{i-1}_n}(x,\theta,F) = \\ &= \int_{t^{i-1}_n}^{t^i_n} \Big(\int_{D\times Z\times\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s,x)) \nabla\dot{\boldsymbol{\varphi}}(s,x)\; d(\boldsymbol{\nu}_n)_s(x,\theta,F) \Big)\, ds, \end{aligned}$$

where $\varepsilon^n(s,x) := \nabla\boldsymbol{\varphi}(s,x) - \nabla\boldsymbol{\varphi}(\tau^n(s),x)$, for every $s \in [0,T]$ and every $x \in D$, and iterating from l to j, we obtain

$$\begin{aligned} &\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n)_{t_2}(x,\theta,F) + \mathrm{Var}_H(\boldsymbol{\mu}_n; t_1, t_2) \le \\ &\le \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\; d(\boldsymbol{\nu}_n)_{t_1}(x,\theta,F) + \int_{\tau^n(t_1)}^{\tau^n(t_2)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\, ds + \\ &+ \int_{\tau^n(t_1)}^{\tau^n(t_2)} \Big(\int_{D\times Z\times\mathbb{R}^{N\times d}} \Big[\frac{\partial W}{\partial F}(\theta, F + \varepsilon^n(s,x)) - \frac{\partial W}{\partial F}(\theta,F)\Big] \nabla\dot{\boldsymbol{\varphi}}(s)\; d(\boldsymbol{\nu}_n)_s(x,\theta,F) \Big)\, ds; \end{aligned} \tag{5.5.24}$$

in particular for $t_1 = 0$ we have

$$\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\ d(\boldsymbol{\nu}_n)_{t_2}(x,\theta,F) + \mathrm{Var}_H(\boldsymbol{\mu}_n;0,t_2) \leq$$
$$\leq \mathcal{W}(z_0,v_0) + \int_0^{\tau^n(t_2)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds + \tag{5.5.25}$$
$$+\int_0^{\tau^n(t_2)} \Big(\int_{D\times Z\times\mathbb{R}^{N\times d}} \Big[\frac{\partial W}{\partial F}(\theta, F+\varepsilon^n(s,x)) - \frac{\partial W}{\partial F}(\theta,F)\Big]\nabla\dot{\boldsymbol{\varphi}}(s)\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)\ ds.$$

From (5.5.25), we can deduce the following a priori estimates on $(\boldsymbol{\nu}_n,\boldsymbol{\mu}_n)$.

Lemma 5.5.3. *There exists a positive constant C, such that*

$$\sup_n \sup_{t\in[0,T]} \int_{D\times Z\times\mathbb{R}^{N\times d}} |(\theta,F)|^{2r}\ d(\boldsymbol{\nu}_n)_t(x,\theta,F) \leq C, \tag{5.5.26}$$
$$\sup_n \mathrm{Var}_H(\boldsymbol{\mu}_n;0,T) \leq C. \tag{5.5.27}$$

Proof. Using the fact that $\int_0^T \|\dot{\boldsymbol{\varphi}}(t)\|_{H^1}\ dt$ is finite, the hypotheses on W and the inequality

$$\sup_{s\in[0,T]} \int_{D\times Z\times\mathbb{R}^{N\times d}} |F|^2\ d(\boldsymbol{\nu}_n)_s(x,\theta,F) < \infty,$$

(since $\boldsymbol{\nu}_n$ are piecewise constant interpolations of Young measures with finite second moments), we can deduce from (5.5.25) that, for n sufficiently large,

$$\int_{D\times Z\times\mathbb{R}^{N\times d}} |F|^2\ d(\boldsymbol{\nu}_n)_t(x,\theta,F) \leq \tag{5.5.28}$$
$$\leq \tilde{C} + \tilde{C} \sup_{s\in[0,T]} \Big(1 + \tilde{c}\int_{D\times Z\times\mathbb{R}^{N\times d}} |F|^2\ d(\boldsymbol{\nu}_n)_s(x,\theta,F)\Big)^{1/2},$$

for suitable positive constants $\tilde{C}$ and $\tilde{c}$ independent of t and n.

Since this can be repeated for every $t \in [0,T]$, we deduce

$$\sup_n \sup_{t\in[0,T]} \int_{D\times Z\times\mathbb{R}^{N\times d}} |(\theta,F)|^2\ d(\boldsymbol{\nu}_n)_t(x,\theta,F) \leq C,. \tag{5.5.29}$$

Inequality (5.5.26) comes now from (5.5.29) and (5.5.4), while inequality (5.5.27) follows from (5.5.29) and (5.5.24). $\square$

Using Lemma 5.5.3 and adapting the proof of Lemma 3.4.5, we can deduce the following discrete version of the energy inequality: for every $t_1 < t_2$ in $[0,T]$

$$\int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\ d(\boldsymbol{\nu}_n)_{t_2}(x,\theta,F) + \mathrm{Var}_H(\boldsymbol{\mu}_n;t_1,t_2) \leq$$
$$\leq \int_{D\times Z\times\mathbb{R}^{N\times d}} W(\theta,F)\ d(\boldsymbol{\nu}_n)_{t_1}(x,\theta,F) + \int_{\tau^n(t_1)}^{\tau^n(t_2)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds + \rho_n, \tag{5.5.30}$$

where $\rho_n \to 0$ as $n \to \infty$.

5.5.3. Passage to the limit. Thanks to (5.5.27), we can apply Helly's Theorem (Theorem 2.3.2) to the sequence $(\boldsymbol{b}_n)_n$ and obtain a subsequence, still indicated with $(\boldsymbol{b}_n)_n$, a subset $\mathcal{T}$ of $[0,T]$, containing 0, with $\mathcal{L}^1([0,T]\setminus\mathcal{T})=0$, and $\boldsymbol{b}\in S_-([0,T],D,q)$, such that, for every finite sequence $t_1<\cdots<t_l$ in $\mathcal{T}$, we have

$$(\boldsymbol{b}_n)^{t_1\dots t_l}_{\alpha_1\dots\alpha_l} \rightharpoonup \boldsymbol{b}^{t_1\dots t_l}_{\alpha_1\dots\alpha_l} \quad L^\infty\text{-weakly*}, \tag{5.5.31}$$

as $n\to\infty$, for every $(\alpha_1,\dots,\alpha_l)\in\mathscr{A}^l_q$. Denote by $\boldsymbol{\mu}$ the element of $SY_-([0,T],D;Z)$ corresponding to $\boldsymbol{b}$.

Let $\mathcal{T}'$ be a dense countable subset of $\mathcal{T}$ containing 0. Thanks to (5.5.26) and Remark 2.3.6, we can find with a diagonalization process a subsequence of $(\boldsymbol{\lambda}_n)_n$, still indicated by $(\boldsymbol{\lambda}_n)_n$, and $\boldsymbol{\lambda}^t=(\boldsymbol{\lambda}^t_\alpha)_\alpha\in Y(D;\mathbb{R}^{N\times d})^q$ for every $t\in\mathcal{T}'$, such that

$$\int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x)|F|^{2r}\,d\boldsymbol{\lambda}^t_\alpha(x,F)\le C, \tag{5.5.32}$$

and

$$(\boldsymbol{b}_n)^t_\alpha(\boldsymbol{\lambda}_n)^t_\alpha \rightharpoonup \boldsymbol{b}^t_\alpha\boldsymbol{\lambda}^t_\alpha \quad 2r\text{-weakly*, as } n\to\infty, \tag{5.5.33}$$

for every $t\in\mathcal{T}'$. Note that the family of coefficients $\boldsymbol{b}$ appearing here is the same as in (5.5.31), because $\pi_{D\times Z}((\boldsymbol{\nu}_n)_t)=(\boldsymbol{\mu}_n)_t$ for every $t\in[0,T]$ and thanks to Remark 2.3.4; moreover, by construction of $(\boldsymbol{\nu}_n,\boldsymbol{\mu}_n)$ we have

$$\boldsymbol{b}^0_\alpha=(\boldsymbol{b}_n)^0_\alpha=1_{D^0_\alpha}, \tag{5.5.34}$$

$$(\boldsymbol{\lambda}^0_\alpha)^x=((\boldsymbol{\lambda}_n)^0_\alpha)^x=\delta_{\nabla v_0(x)} \quad \text{for a.e. } x\in D^0_\alpha, \tag{5.5.35}$$

$$\tag{5.5.36}$$

where $D^0_\alpha:=\{x\in D\ :\ z_0(x)=\theta_\alpha\}$.

For every $t\in\mathcal{T}\setminus\mathcal{T}'$, let us choose an increasing sequence of integers n^t_k, possibly depending on t, such that

$$\limsup_n\langle\boldsymbol{\sigma}_n(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle=\lim_k\langle\boldsymbol{\sigma}_{n^t_k}(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle \tag{5.5.37}$$

(this choice is crucial in order to apply the argument in [**13**, Section 7]). Again by (5.5.26) and Remark 2.3.6, we can extract a further subsequence, still denoted by $(\boldsymbol{\lambda}_{n^t_k})_k$, satisfying (5.5.37) and such that there exists $\boldsymbol{\lambda}^t\in Y(D;\mathbb{R}^{N\times d})^q$ with

$$\int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}^t_\alpha(x)|F|^{2r}\,d\boldsymbol{\lambda}^t_\alpha(x,F)\le C, \tag{5.5.38}$$

$$(\boldsymbol{b}_{n^t_k})^t_\alpha(\boldsymbol{\lambda}_{n^t_k})^t_\alpha \rightharpoonup \boldsymbol{b}^t_\alpha\boldsymbol{\lambda}^t_\alpha \quad 2r\text{-weakly*, as } k\to\infty. \tag{5.5.39}$$

Note that, thanks to **(W.4)**, we have

$$\limsup_n\langle\boldsymbol{\sigma}_n(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle=\langle\boldsymbol{\sigma}(t),\nabla\dot{\boldsymbol{\varphi}}(t)\rangle, \tag{5.5.40}$$

where

$$\boldsymbol{\sigma}(t,x):=\sum_\alpha\boldsymbol{b}^t_\alpha(x)\int_{\mathbb{R}^{N\times d}}\tfrac{\partial W}{\partial F}(\theta_\alpha,F)\,d(\boldsymbol{\lambda}^t_\alpha)^x(F),$$

for every $t \in \mathcal{T}$. This implies that the map (5.4.1) is measurable on $[0,T]$; moreover for every $t \in \mathcal{T}'$ we have

$$\limsup_n \langle \boldsymbol{\sigma}_n(t), \nabla \dot{\boldsymbol{\varphi}}(t)\rangle = \lim_n \langle \boldsymbol{\sigma}_n(t), \nabla \dot{\boldsymbol{\varphi}}(t)\rangle.$$

The family $\boldsymbol{\nu}$ will denote the element of $Y^{2r}(D; Z \times \mathbb{R}^{N\times d})^{\mathcal{T}}$ corresponding to $(\boldsymbol{b}, \boldsymbol{\lambda})$. Let $t \in [0,T] \setminus \mathcal{T}$, and fix a sequence s_j in $\mathcal{T}$ converging to t with $s_j < t$; by (5.5.32), and (5.5.38), we have

$$\sup_j \int_D \boldsymbol{b}_\alpha^{s_j}(x)\Big(\int_{\mathbb{R}^{N\times d}} |F|^{2r}\; d(\boldsymbol{\lambda}_\alpha^{s_j})^x(F)\Big)\; dx \le C,$$

for every j; again by Remark 2.3.6, we can find a subsequence, not relabeled, and $\boldsymbol{\lambda}^t \in Y(D; \mathbb{R}^{N\times d})$, such that

$$\int_{D\times\mathbb{R}^{N\times d}} \boldsymbol{b}_\alpha^t(x)|F|^{2r}\; d\boldsymbol{\lambda}_\alpha^t(x,F) \le C, \tag{5.5.41}$$

and

$$\boldsymbol{b}_\alpha^{s_j}\boldsymbol{\lambda}_\alpha^{s_j} \rightharpoonup \boldsymbol{b}_\alpha^t\boldsymbol{\lambda}_\alpha^t \quad 2r\text{-weakly*}, \;\text{ as } j \to \infty. \tag{5.5.42}$$

Note that, since $\pi_{D\times Z}(\boldsymbol{\nu}_t) = \boldsymbol{\mu}_t$ for every $t \in \mathcal{T}$, the left continuity of $\boldsymbol{b}$ defined in (5.5.31 ensures that the family of coefficients appearing in (5.5.42) is the same as in (5.5.31).

In this way we have defined $\boldsymbol{\lambda} \in (Y(D;\mathbb{R}^{N\times d})^q)^{[0,T]}$, and consequently $\boldsymbol{\nu} \in Y^{2r}(D; Z \times \mathbb{R}^{N\times d})^{[0,T]}$. It can be shown that $(\boldsymbol{b}, \boldsymbol{\lambda}) \in Ad([0,T], q, \boldsymbol{\varphi})$ using Lemma 5.3.3 and adapting the argument in Section 3.4.3.

By construction $(\boldsymbol{b}, \boldsymbol{\lambda})$ satisfies (ev0).

5.5.4. Stability. Fix n and $i = 1, \dots, k(n)$. Let

$$\begin{aligned} M\colon D &\to \mathbb{M}^{q\times q}_{St} \\ x &\mapsto (M_{\beta\alpha}(x))_{\beta\alpha}, \end{aligned}$$

be a measurable map, and let $\tilde u \in H_0^1(D;\mathbb{R}^N)$. We define $(\tilde{\boldsymbol{\nu}}_n^i, \tilde{\boldsymbol{\mu}}_n^i) \in Y^2(D; Z\times\mathbb{R}^{N\times d})^{\{t_n^0,\dots,t_n^i\}} \times SY(\{t_n^0, \dots, t_n^i\}, D, Z)$ by

$$\begin{aligned} (\tilde{\boldsymbol{\nu}}_n^i)_{t_n^j} &:= (\boldsymbol{\nu}_n^i)_{t_n^j} \quad \text{if } j < i \\ (\tilde{\boldsymbol{\nu}}_n^i)_{t_n^i} &:= \sum_{\alpha,\beta} M_{\beta\alpha}(\boldsymbol{b}_n^i)_\alpha^{t_n^i}(\delta_{\theta_\beta} \otimes \tau_{\nabla\tilde u}(\boldsymbol{\lambda}_n^i)_\alpha^{t_n^i}), \\ (\tilde{\boldsymbol{\mu}}_n^i)_{t_n^0\dots t_n^i} &:= \sum_{\alpha,\beta} M_{\beta\alpha}(\boldsymbol{b}_n^i)^{t_n^0\dots t_n^i}_{\alpha_0\dots\alpha_{i-1}\alpha}\boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_{i-1}},\theta_\beta)}, \end{aligned}$$

where $\tau_{\nabla\tilde u}\colon D\times\mathbb{R}^{N\times d} \to D\times\mathbb{R}^{N\times d}$ is the map sending (x,F) into $(x, F + \nabla\tilde u(x))$.

Lemma 5.5.4. *The pair $(\tilde{\boldsymbol{\nu}}_n^i, \tilde{\boldsymbol{\mu}}_n^i)$ is in $AY(\{t_n^0, \dots, t_n^i\}, Z, \boldsymbol{\varphi})$.*

Proof. Consider $(\boldsymbol{\nu}_n^i, \boldsymbol{\mu}_n^i) \in AY(\{t_n^0, \dots, t_n^i\}, Z, \boldsymbol{\varphi})$: for every $j = 0, \dots, i$, there exist a sequence $(v_k^j)_k$ contained in $\boldsymbol{\varphi}(t_n^j) + H_0^1(D;\mathbb{R}^N)$, and a sequence $((D_\alpha^j)_k)_\alpha$, indexed by k, of measurable partitions of D, such that

(1) we have
$$\sum_{\alpha_0,\dots,\alpha_i} 1_{(D^0_{\alpha_0})_k}\cdot\dots\cdot 1_{(D^i_{\alpha_i})_k}\boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_i})} \rightharpoonup (\boldsymbol{\mu}^i_n)_{t^0_n\dots t^i_n} \quad \text{weakly*},$$
as $k\to\infty$;

(2) for every $j=0,\dots,i$, there exists a subsequence $(k^j_l)_l$, possibly dependent on j, such that
$$\sum_{\alpha=1}^q 1_{(D^j_\alpha)_{k^j_l}}\boldsymbol{\delta}_{(\theta_\alpha,\nabla v^j_{k^j_l})} \rightharpoonup (\boldsymbol{\nu}^i_n)_{t^j_n} \quad \text{2-weakly*},$$
as $l\to\infty$.

In particular, by Lemma 1.3.6 this implies that
$$\sum_{\alpha_0,\dots,\alpha_{i-1},\alpha,\beta} M_{\beta\alpha}1_{(D^0_{\alpha_0})_k}\cdot\dots\cdot 1_{(D^{i-1}_{\alpha_{i-1}})_k}\cdot 1_{(D^i_\alpha)_k}\boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_{i-1}},\theta_\beta)} \rightharpoonup (\tilde{\boldsymbol{\mu}}^i_n)_{t^0_n\dots t^i_n} \quad \text{weakly*, as } k\to\infty;$$
$$\sum_{\alpha,\beta} M_{\beta\alpha}1_{(D^i_\alpha)_{k^i_l}}\boldsymbol{\delta}_{(\theta_\beta,\nabla v^i_{k^i_l}+\nabla\tilde u)} \rightharpoonup (\tilde{\boldsymbol{\nu}}^i_n)_{t^i_n} \quad \text{2-weakly*, as } l\to\infty;$$
$$\sum_{\alpha=1}^q 1_{(D^j_\alpha)_{k^j_l}}\boldsymbol{\delta}_{(\theta_\alpha,\nabla v^j_{k^j_l})} \rightharpoonup (\tilde{\boldsymbol{\nu}}^i_n)_{t^j_n} \quad \text{2-weakly*, as } l\to\infty, \text{ for every } j<i.$$

Thanks to Lemma 5.3.4, the pair $(\tilde{\boldsymbol{\nu}}^k,\tilde{\boldsymbol{\mu}}^k)\in Y^2(D;Z\times\mathbb{R}^{N\times d})^{\{t^0_n,\dots,t^i_n\}}\times SY^2(\{t^0_n,\dots,t^i_n\},D;Z)$, defined by
$$(\tilde{\boldsymbol{\nu}}^k)_{t^i_n} := \sum_{\alpha,\beta} M_{\beta\alpha}1_{(D^i_\alpha)_k}\boldsymbol{\delta}_{(\theta_\beta,\nabla v^i_k+\nabla\tilde u)},$$
$$(\tilde{\boldsymbol{\nu}}^k)_{t^j_n} := \sum_{\alpha=1}^q 1_{(D^j_\alpha)_k}\boldsymbol{\delta}_{(\theta_\alpha,\nabla v^j_k)} \quad \text{for every } j<i,$$
$$(\tilde{\boldsymbol{\mu}}^k)_{t^0_n\dots t^i_n} := \sum_{\alpha_0,\dots,\alpha_{i-1},\alpha,\beta} M_{\beta\alpha}1_{(D^0_{\alpha_0})_k}\cdot\dots\cdot 1_{(D^{i-1}_{\alpha_{i-1}})_k}\cdot 1_{(D^i_\alpha)_k}\boldsymbol{\delta}_{(\theta_{\alpha_0},\dots,\theta_{\alpha_{i-1}},\theta_\beta)},$$
is an element of $AY(\{t^0_n,\dots,t^i_n\},Z,\boldsymbol{\varphi})$, for every k. Therefore, the thesis can be deduced using Lemma 3.3.6. $\square$

Set
$$(\tilde{\boldsymbol{b}}^i_n)^{t^0_n\dots t^{i-1}_n t^i_n}_{\alpha_0\dots\alpha_{i-1}\beta} := \sum_\alpha M_{\beta\alpha}(\boldsymbol{b}^i_n)^{t^0_n\dots t^{i-1}_n t^i_n}_{\alpha_0\dots\alpha_{i-1}\alpha},$$
$$(\tilde{\boldsymbol{\lambda}}^i_n)^{t^j_n}_\beta := (\boldsymbol{\lambda}^i_n)^{t^i_n}_\beta \quad \text{for every } j<i,$$
$$((\tilde{\boldsymbol{\lambda}}^i_n)^{t^i_n}_\beta)^x := \frac{\sum_\alpha M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\tilde\tau_{\nabla\tilde u(x)}(((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x)}{\sum_\alpha M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)} \quad \text{if } \sum_\alpha M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)>0$$
for a.e. $x\in D$, for every β, and every $(\alpha_0,\dots,\alpha_{i-1})\in\mathscr{A}^i_q$, where $\tilde\tau_{\nabla\tilde u(x)}\colon\mathbb{R}^{N\times d}\to\mathbb{R}^{N\times d}$ is the map defined by $F\mapsto F+\nabla\tilde u(x)$, for a.e. $x\in D$; since $(\tilde{\boldsymbol{b}}^i_n,\tilde{\boldsymbol{\lambda}}^i_n)$ is the element

corresponding to $(\tilde{\boldsymbol{\nu}}^i_n, \tilde{\boldsymbol{\mu}}^i_n)$, we immediate deduce from Lemma 5.5.4 that $(\tilde{\boldsymbol{b}}^i_n, \tilde{\boldsymbol{\lambda}}^i_n)$ belongs to $A^i_n(\boldsymbol{b}^{i-1}_n, \boldsymbol{\lambda}^{i-1}_n)$. The minimizing property of $(\boldsymbol{b}^i_n, \boldsymbol{\lambda}^i_n)$ implies that

$$\sum_\alpha \int_D (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F)\Big)\ dx + \\ + \sum_{\alpha\gamma} H(\theta_\alpha,\theta_\gamma)\int_D (\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx \leq \\ \leq \sum_\alpha \int_D (\tilde{\boldsymbol{b}}^i_n)^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\tilde{\boldsymbol{\lambda}}^i_n)^{t^i_n}_\alpha)^x(F)\Big)\ dx + \\ + \sum_{\beta\gamma} H(\theta_\beta,\theta_\gamma)\int_D (\tilde{\boldsymbol{b}}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\beta}(x)\ dx;$$

in other words

$$\sum_\alpha \int_D (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F)\Big)\ dx \leq \\ \leq \sum_{\alpha\beta} \int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta, F+\nabla\tilde{u})\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F)\Big)\ dx + \quad (5.5.43) \\ \leq \sum_{\alpha\gamma\beta} H(\theta_\beta,\theta_\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx - \sum_{\alpha\gamma} H(\theta_\alpha,\theta_\gamma)\int_D (\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx.$$

Since $\sum_\beta M_{\beta\alpha}(x) = 1$ for a.e. $x \in D$ and every α, we can deduce, using the triangular inequality, that

$$\sum_{\alpha\beta\gamma} H(\theta_\beta,\theta_\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx - \sum_{\alpha\gamma} H(\theta_\alpha,\theta_\gamma)\int_D (\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx = \\ \leq \sum_{\alpha\beta\gamma} H(\theta_\beta,\theta_\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx - \sum_{\alpha\beta\gamma} H(\theta_\alpha,\theta_\gamma)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx \leq \\ \leq \sum_{\alpha\beta} H(\theta_\beta,\theta_\alpha)\int_D M_{\beta\alpha}(x)\sum_\gamma (\boldsymbol{b}^i_n)^{t^{i-1}_n t^i_n}_{\gamma\alpha}(x)\ dx = \sum_{\alpha\beta} H(\theta_\beta,\theta_\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\ dx.$$

Hence we deduce from (5.5.43) that

$$\sum_\alpha \int_D (\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F)\Big)\ dx \leq \\ \leq \sum_{\alpha\beta} \int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta, F+\nabla\tilde{u}(x))\ d((\boldsymbol{\lambda}^i_n)^{t^i_n}_\alpha)^x(F)\Big)\ dx + \\ + \sum_{\alpha\beta} H(\theta_\beta,\theta_\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}^i_n)^{t^i_n}_\alpha(x)\ dx,$$

for every n and $i = 1, \dots, k(n)$; we can rewrite the previous inequality in the following form

$$\sum_\alpha \int_D (\boldsymbol{b}_n)^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}_n)^t_\alpha)^x(F)\Big)\ dx \le$$
$$\le \sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n)^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta, F+\nabla\tilde{u}(x))\ d((\boldsymbol{\lambda}_n)^t_\alpha)^x(F)\ dx + \tag{5.5.44}$$
$$+\sum_{\alpha\beta} H(\theta_\beta,\theta_\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n)^t_\alpha(x)\ dx,$$

for every $t \in \mathcal{T}\setminus\{0\}$ and every n. From (5.5.31) we can deduce that

$$\sum_{\alpha\beta} H(\theta_\beta,\theta_\alpha)\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_n)^t_\alpha(x)\ dx \to \sum_{\alpha\beta} H(\theta_\beta,\theta_\alpha)\int_D M_{\beta\alpha}(x)\boldsymbol{b}^t_\alpha(x)\ dx, \tag{5.5.45}$$

as $n\to\infty$, for every $t\in\mathcal{T}\setminus\{0\}$. Consider $(\bar{\boldsymbol{\nu}}_n)_t := \sum_{\alpha\beta} M_{\beta\alpha}(\boldsymbol{b}_n)^t_\alpha(\delta_{\theta_\beta}\otimes(\boldsymbol{\lambda}_n)^t_\alpha)$, for every $t\in(0,T]$; we have

$$\sup_n \sup_{t\in(0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}} |(\theta,F)|^{2r}\ d(\bar{\boldsymbol{\nu}}_n)_t(x,\theta,F) =$$
$$=\sup_n\sup_{t\in(0,T]}\int_D \sum_{\alpha\beta} M_{\beta\alpha}(x)(\boldsymbol{b}_n)^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} |(\theta_\beta,F)|^{2r}\ d((\boldsymbol{\lambda}_n)^t_\alpha)^x(F)\Big)\ dx \le$$
$$\le \sup_\alpha|\theta_\alpha|^{2r} + q\sup_n\sup_{t\in(0,T]}\int_D\sum_\alpha(\boldsymbol{b}_n)^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}}|(\theta_\alpha,F)|^{2r}\ d((\boldsymbol{\lambda}_n)^t_\alpha)^x(F)\Big)\ dx =$$
$$= K + q\sup_n\sup_{t\in(0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|(\theta,F)|^{2r}\ d(\boldsymbol{\nu}_n)_t(x,\theta,F),$$

for $K := \sup_\alpha|\theta_\alpha|^{2r}$; therefore we can deduce from (5.5.26) that

$$\sup_n\sup_{t\in[0,T]}\int_{D\times Z\times\mathbb{R}^{N\times d}}|(\theta,F)|^{2r}\ d(\bar{\boldsymbol{\nu}}_n)_t(x,\theta,F)\le K+qC.$$

In particular for every $t\in\mathcal{T}\setminus\{0\}$, we deduce from (5.5.33) and (5.5.39) that

$$(\bar{\boldsymbol{\nu}}_{n^t_k})_t \rightharpoonup \bar{\boldsymbol{\nu}}_t := \sum_{\alpha\beta} M_{\beta\alpha}\boldsymbol{b}^t_\alpha(\delta_{\theta_\beta}\otimes\boldsymbol{\lambda}^t_\alpha)\quad 2r\text{-weakly*}, \tag{5.5.46}$$

as $k\to\infty$, where $(\boldsymbol{b},\boldsymbol{\lambda})$ is the pair defined by (5.5.31), (5.5.33), and (5.5.39). Since $|W(\theta,F+\nabla\tilde{u}(x))|\le C(1+|\nabla\tilde{u}(x)|^2)+C|F|^2$, we can use Lemma 1.3.5 to deduce from (5.5.46) that

$$\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)(\boldsymbol{b}_{n^t_k})^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta,F+\nabla\tilde{u}(x))\ d((\boldsymbol{\lambda}_{n^t_k})^t_\alpha)^x(F)\Big)\ dx \to$$
$$\to\sum_{\alpha\beta}\int_D M_{\beta\alpha}(x)\boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta,F+\nabla\tilde{u}(x))\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\Big)\ dx, \tag{5.5.47}$$

as $k \to \infty$. Analogously, we deduce that

$$\sum_\alpha \int_D (\boldsymbol{b}_{n_k^t})_\alpha^t(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}_{n_k^t})_\alpha^t)^x(F)\Big)\ dx \to$$
$$\to \sum_\alpha \int_D \boldsymbol{b}_\alpha^t(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_\alpha^t)^x(F)\Big)\ dx, \tag{5.5.48}$$

as $k \to \infty$; therefore using (5.5.44), (5.5.48), (5.5.47), and (5.5.45), we can deduce immediately (ev1), for every $t \in \mathcal{T} \setminus \{0\}$, while for $t = 0$ it is an obvious consequence of (ev0) and the hypothesis on the initial datum (z_0, v_0). For $t \in [0, T] \setminus \mathcal{T}$, (ev1) can be easily proved using (5.5.42) and (ev1) for $t \in \mathcal{T}$, as in Section 3.4.3.

5.5.5. Energy inequality. Let $t_1 < t_2$ be two time instants in $\mathcal{T}$; choose any $t \in (t_1, t_2) \cap \mathcal{T}'$. Let $(\boldsymbol{\lambda}_{n_k^1})_k$ and $(\boldsymbol{\lambda}_{n_k^2})_k$ be two sequences satisfying (5.5.37) and (5.5.39) for t_1 and t_2 respectively. We have

$$\sum_\alpha \int_D \boldsymbol{b}_\alpha^{t_2}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_\alpha^{t_2})^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) \leq$$
$$\leq \liminf_k \Big[\sum_\alpha \int_D (\boldsymbol{b}_{n_k^2})_\alpha^{t_2}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}_{n_k^2})_\alpha^{t_2})^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}_{n_k^2}; t, t_2)\Big] +$$
$$+\mathrm{Diss}_H(\boldsymbol{b}; t_1, t);$$

using (5.5.30), we can deduce that

$$\sum_\alpha \int_D \boldsymbol{b}_\alpha^{t_2}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_\alpha^{t_2})^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) \leq$$
$$\leq \sum_\alpha \int_D \boldsymbol{b}_\alpha^{t}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_\alpha^{t})^x(F)\Big)\ dx +$$
$$+ \limsup_n \int_{\tau^n(t)}^{\tau^n(t_2)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t);$$

since $\sup_t \sup_n \|\boldsymbol{\sigma}_n(t)\|_2$ is finite, we have by Fatou Lemma

$$\limsup_n \int_{\tau^n(t)}^{\tau^n(t_2)} \langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds \leq \int_0^T \limsup_n [1_{[\tau^n(t), \tau^n(t_2)]}(s)\langle \boldsymbol{\sigma}_n(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle]\ ds =$$
$$= \int_t^{t_2} \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds.$$

Hence we have

$$\sum_\alpha \int_D \boldsymbol{b}_\alpha^{t_2}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_\alpha^{t_2})^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) \leq$$
$$\leq \sum_\alpha \int_D \boldsymbol{b}_\alpha^{t}(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}_\alpha^{t})^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t) + \tag{5.5.49}$$
$$+ \int_t^{t_2} \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds;$$

we now apply the same argument to the time interval $[t_1, t]$, i.e.,

$$\begin{aligned}
&\sum_\alpha \int_D \boldsymbol{b}^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^t_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t) \leq \\
&\leq \liminf_n \Big[\sum_\alpha \int_D (\boldsymbol{b}_n)^t_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}_n)^t_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}_n; t_1, t)\Big] \leq \\
&\leq \liminf_k \Big[\sum_\alpha \int_D (\boldsymbol{b}_{n^1_k})^{t_1}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d((\boldsymbol{\lambda}_{n^1_k})^{t_1}_\alpha)^x(F)\Big)\ dx + \qquad (5.5.50)\\
&\qquad + \int_{\tau^{n^1_k}(t_1)}^{\tau^{n^1_k}(t)} \langle \boldsymbol{\sigma}_{n^1_k}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds\Big] \leq \\
&\leq \sum_\alpha \int_D \boldsymbol{b}^{t_1}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^{t_1}_\alpha)^x(F)\Big)\ dx + \int_{t_1}^t \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds.
\end{aligned}$$

From (5.5.49) and (5.5.50) we obtain

$$\begin{aligned}
&\sum_\alpha \int_D \boldsymbol{b}^{t_2}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^{t_2}_\alpha)^x(F)\Big)\ dx + \mathrm{Diss}_H(\boldsymbol{b}; t_1, t_2) \leq \\
&\leq \sum_\alpha \int_D \boldsymbol{b}^{t_1}_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d(\boldsymbol{\lambda}^{t_1}_\alpha)^x(F)\Big)\ dx + \int_{t_1}^{t_2} \langle \boldsymbol{\sigma}(s), \nabla\dot{\boldsymbol{\varphi}}(s)\rangle\ ds.
\end{aligned}$$

We have proved (ev2) for $t_1 < t_2$ in $\mathcal{T}$. Using (5.5.42) and the left continuity of $\boldsymbol{b}$, the same argument as in Section 3.4.3 proves (ev2) for $t_1 < t_2$ in $[0, T]$.

5.6. Euler conditions

In this section we derive the Euler equations for the partial-global stability condition.

THEOREM 5.6.1. *Let $(b, \lambda) \in L^\infty(D; [0,1])^q \times Y(D; \mathbb{R}^{N\times d})^q$ satisfy (2.3.2) and (2.3.10) with $p = 2$. Assume that (b, λ) satisfies*

$$\begin{aligned}
&\sum_{\alpha=1}^q \int_D b_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\alpha, F)\ d\lambda^x_\alpha(F)\Big)\ dx \leq \\
&\leq \sum_{\alpha,\beta=1}^q \int_D M_{\beta\alpha}(x) b_\alpha(x)\Big(\int_{\mathbb{R}^{N\times d}} W(\theta_\beta, F + \nabla\tilde{u}(x))\ d\lambda^x_\alpha(F)\Big)\ dx + \qquad (5.6.1)\\
&\qquad + \sum_{\alpha,\beta=1}^q H(\theta_\beta, \theta_\alpha) \int_D M_{\beta\alpha}(x) b_\alpha(x)\ dx,
\end{aligned}$$

for every $\tilde{u} \in H^1_0(D; \mathbb{R}^N)$ and every measurable map $M\colon D \to \mathbb{M}^{q\times q}_{St}$, and denote by σ the stress, i.e.,

$$\sigma(x) := \sum_{\alpha=1}^q b_\alpha(x) \int_{\mathbb{R}^{N\times d}} \frac{\partial W}{\partial F}(\theta_\alpha, F)\ d\lambda^x_\alpha(F), \quad \textit{for a.e. } x \in D.$$

Then the following conditions are satisfied:

$(\mathrm{ec})_1$ equilibrium condition*:* $\mathrm{div}\sigma(t) = 0$*;*

$(\mathrm{ec})_2$ optimality of active phases*: for every* $\alpha, \beta = 1, \dots, q$ *and every* $t \in [0,T]$*, we have*

$$\int_{\mathbb{R}^{N\times d}} [W(\theta_\alpha, F) - W(\theta_\beta, F)]\; d\lambda_\alpha^x(F) \le H(\theta_\beta, \theta_\alpha),$$

for a.e. $x \in D$ *with* $b_\alpha(x) > 0$.

Remark 5.6.2. We say that a phase θ_α is *active* at x if $b_\alpha(x) > 0$. Hence the condition $(\mathrm{ec})_2$ can be rephrased as follows: if θ_α is active at x, then θ_α is a minimizer over Z of the functional

$$\theta \mapsto \int_{\mathbb{R}^{N\times d}} W(\theta, F)\; d\lambda_\alpha^x(F) + H(\theta, \theta_\alpha)$$

This is the reason why we call $(\mathrm{ec})_2$ optimality of active phases.

Remark 5.6.3. Note that from $(\mathrm{ec})_2$ it descends immediately that

$$\sum_{\alpha,\beta} M_{\beta\alpha} b_\alpha(x) \int_{\mathbb{R}^{N\times d}} [W(\theta_\alpha, F) - W(\theta_\beta, F) - H(\theta_\beta, \theta_\alpha)]\; d\lambda_\alpha^x(F) \le 0,$$

for a.e. $x \in D$ and every stochastic matrix $M \in \mathbb{M}^{q\times q}_{St}$.

Proof of Theorem 5.6.1. Let (b, λ) satisfy the prescribed hypotheses. Choosing in (5.6.1) the map M associating to every $x \in D$ the identity matrix $\mathbb{I}$, we obtain

$$\sum_\alpha \int_D b_\alpha(x) \Big[\int_{\mathbb{R}^{N\times d}} \big[W(\theta_\alpha, F + \nabla\tilde{u}(x)) - W(\theta_\alpha, F)\big]\; d\lambda_\alpha)^x(F) \Big]\; dx \ge 0,$$

for every $\tilde{u} \in H^1_0(D; \mathbb{R}^N)$, which implies immediately $(\mathrm{ec})_1$.

Let us denote by $(e_\gamma)_{\gamma=1}^q$ the canonical basis of the vector space $\mathbb{R}^q$. Fixed α, β in $\{1, \dots, q\}$, define $\bar{M} \in \mathbb{M}^{q\times q}_{St}$ by

$$\bar{M} e_\gamma = e_\gamma \quad \text{for every } \gamma \ne \alpha$$
$$\bar{M} e_\alpha = e_\beta.$$

Let us choose now in (5.6.1) $\tilde{u} = 0$ and $M := \mathbb{I}(1 - 1_A) + \bar{M} 1_A$, for any measurable subset A of D: we obtain

$$\int_A b_\alpha(x) \Big[\int_{\mathbb{R}^{N\times d}} \big[W(\theta_\alpha, F) - W(\theta_\beta, F)\big]\; d\lambda_\alpha^x(F) \Big]\; dx \le$$
$$\le \int_A H(\theta_\beta, \theta_\alpha) b_\alpha(x)\; dx; \tag{5.6.2}$$

By the free choice of A among all measurable subsets of D, from (5.6.2) we deduce immediately $(\mathrm{ec})_2$. □

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Ringraziamenti

È con profonda nostalgia che mi avvicino al mio commiato da Trieste e dalla SISSA: questi quattro anni sono stati per me un'esperienza bellissima, resa unica dalle persone che ho avuto modo di conoscere in questo periodo e alle quali vanno il mio affetto e la mia riconoscenza.

Primo tra tutti, desidero ringraziare il mio relatore, Prof. Gianni Dal Maso: la sua profonda conoscenza della matematica e la solidità del suo ragionamento non finiscono mai di stupirmi e affascinarmi. La possibilità di lavorare sotto la sua guida e di assistere alle sue lezioni è stata per me veramente preziosa e stimolante. Non posso non menzionare l'infinita pazienza, disponibilità e gentilezza che ha sempre dimostrato nei miei riguardi, e che mi hanno permesso di lavorare con la maggiore serenità possibile.

Desidero esprimere la mia gratitudine alla Prof. Giulia Treu per l'interesse dimostrato nei miei confronti, per il suo costante sostegno e per i preziosi consigli, nonché per avermi dato la possibilità di fare le mie prime esperienze didattiche.

Vorrei ringraziare anche tutti i Professori del Settore di Analisi Funzionale: i corsi che ho seguito durante il dottorato sono stati per me davvero formativi e stimolanti.

In particolare sono grata a Maria Giovanna e Massimiliano, per la loro disponibilità e il loro sostegno, che mi sono stati di grande aiuto dal punto di vista umano e lavorativo.

Con grandissimo affetto desidero ringraziare i compagni e gli amici che hanno condiviso con me questa esperienza:

innanzitutto Giuliano e Luca, che sono stati a tutti gli effetti la mia famiglia triestina; la loro costante presenza e il loro sincero affetto hanno per me un valore inestimabile,

Caterina, per essermi stata così vicina in questi ultimi mesi e per avermi fatto rimettere in gioco molte opinioni che credevo definitive,

Maria, per la preziosa amicizia su cui posso sempre contare nonostante la distanza,

Marietta, per il suo trascinante entusiasmo, ma soprattutto per avermi mostrato il lato "mariettoso" della vita,

Paola, per aver condiviso con me pene e dolori dei rapporti a distanza, e per la costante sincerità e schiettezza,

Alessio e Francesca C., per il loro autentico pessimismo che supera anche il mio, per la loro ironia, e per avermi aiutato a sdrammatizzare nei momenti più difficili,

i miei ex-coinquilini Daniel e Riccardo, per aver condiviso con me gioie e dolori di Scala al Belvedere 1, e Alexey, FFR, Fabrizio, Francesco B., Francesco P., Francesco S., Laura, Luca H., Lucio, Marita, Mattia, Nicola, Paolo, Tatjana, Valeria e tutti gli altri.

Un ringraziamento speciale va alle tante persone che lavorano a vario titolo alla SISSA e che mi hanno accolto ogni giorno con il loro genuino sorriso: le signore alla reception, Cristina e Michelle, Barbara Corzani, lo staff del bar e della mensa, la segreteria studenti e l'help desk.

Trieste rimarrà nel mio cuore ovunque andrò, per la sua scontrosa bellezza e per l'autenticità della sua gente; porterò con me la sensazione di grande serenità che mi accompagna quando, tornando a casa dalla SISSA la sera, mi accolgono le mille luci del porto e il profumo del mare. Desidero dedicare questo lavoro di tesi a questa città, che per me sarà sempre casa.

Sono molto felice di avere la possibilità di ringraziare la mia famiglia per tutto l'amore che ha per me: la Franca, per la sua grinta nell'affrontare la vita, che mi è stata più volte di esempio e appiglio, la nonna Bruna, per tutte le sue amorevoli attenzioni e la sua saggezza. Non ringrazierò mai abbastanza i miei genitori, che mi hanno sostenuto, appoggiato e incoraggiato ogni giorno di questi quattro anni; la distanza non ha loro impedito di capire sempre ogni mio stato d'animo e di starmi vicino specialmente nei momenti difficili. È per me di grande conforto e fonte di speranza per il futuro vedere che entrambi hanno realizzato, in modo diverso ma in ugual misura, tutto quello che per me conta nella vita.

Infine, desidero ringraziare dal profondo del mio cuore Francesco, che riunisce in un solo essere tutte le qualità (e i difetti) che io apprezzo e amo in una persona; nulla al mondo può rendermi più felice che averlo accanto. Il suo amore per me è un miracolo che si ripete ogni giorno.